CHAPTER 2
REVIEW OF LITERATURE

2.1 HISTORICAL OVERVIEW OF SOLITONS.

The first recorded solitary wave [91] was observed in the 1834 when a young engineer named John Scott Russell was hired for a summer job to investigate how to improve the efficiency of designs for barges that were designated to ply canals particularly the Union Canal near Edinburgh, Scotland. One August day, the tow rope that was connecting the mules to the barge broke and the barge suddenly stopped—but the mass of water in front of its blunt prow "... rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well defined heap of water, which continued its course along the channel without change of form or diminution of speed." [109]

Russell pursued this serendipitous observation and "... followed it [the launched 'Wave of Translation'] on horseback, and overtook it still rolling on at a rate of some eight or nine miles per hour, preserving its original form some thirty feet long and a foot to a foot and a half in height." He then conducted controlled laboratory experiments using a wave tank and quantified the phenomenon in an 1844 publication [109].

He demonstrated four facts:

1. The solitary waves that he observed had a hyperbolic secant shape.
2. A sufficiently large initial mass of water can produce two or more independent near-solitary waves that separate in time.
3. Solitary waves can cross each other "without change of any kind".
4. In shallow water channel of height 'h' a solitary wave of amplitude 'A' travels at a speed of $\sqrt{\left[g\left(A+h\right)\right]}$ where 'g' is the gravitational acceleration. That is, larger-amplitude waves move faster than smaller ones—a nonlinear effect.

In 1895, Dutch physicist Diederick Korteweg and his student Gustav de Vries (KdV) derived a nonlinear partial differential equation

$$\phi_t + \phi_{xxx} + 6\phi\phi_x = 0 \qquad (1)$$

that now bears their name. Korteweg and de Vries argued that the KdV equation (1) could describe Russell's experiments. Equation (1) shows that the rate of change of the wave's height in time is governed by the sum of two terms: a nonlinear one (the amplitude effect) and a dispersive one (the effect that causes waves of different wavelengths to travel with different velocities). Korteweg and de Vries found a periodic solution in addition to a solitary-wave solution that resembled the wave that Russell had followed. These solutions arose as a result of a balance between nonlinearity and dispersion.

Their work and Russell's observations fell into obscurity and were ignored by mathematicians, physicists, and engineers studying water waves until 1965 when Norman Zabusky and Martin Kruskal published their numerical solutions of the KdV equation (and invented the term "soliton") [131]. Kruskal derived (1) as an asymptotic (continuum) description of oscillations of unidirectional waves propagating on the "cubic" Fermi–Pasta–Ulam (FPU) nonlinear lattice [47,103,129]. Meanwhile, Morikazu Toda became the first to discover a soliton in a discrete, integrable system (the system is now referred to as the Toda lattice) [124].

In 1965, Gary Deem, Zabusky, and Kruskal [8] produced films that showed interacting solitary waves in an FPU lattice, the KdV equation, and a modified KdV equation; see the discussion in the review article [132]. We depict the dynamics of solitons in the KdV equation in the space-time diagram of Figure 1. Robert Miura recognized the significance of this result and found an exact transformation between this modified KdV equation and equation (1) [93]. This awakened the mathematical study of solitons, as Clifford Gardner, John Greene, Martin Kruskal, and Robert Miura in 1967 were able to solve the initial-value problem for the KdV equation by introducing the inverse scattering method [94 ,55, 52] providing an appropriate notion of integrability for continuum frameworks. Vladimir Zakharov and Alexei Borisovich Shabat generalized the inverse scattering method in 1972 when they solved the nonlinear Schrödinger (NLS) equation, another model nonlinear PDE, demonstrating both its integrability and the existence of soliton solutions. In 1973, Mark Ablowitz, David Kaup, Alan Newell, and Harvey Segur demonstrated the existence of soliton solutions (and proved the integrability) of several other nonlinear PDEs, including the sine–Gordon equation (which was already known to be integrable based on Albert Backlünd's 19th century investigations of surfaces with constant negative Gaussian curvature). Other researchers have subsequently derived other integrable PDEs (in both one and multiple spatial dimensions) and constructed accompanying soliton solutions. As the Kadomtsev–Petviashvili (KP) equation illustrates (see Section 3), one needs to be more nuanced as to what constitutes a "soliton" in multiple spatial dimensions. When studying solitary waves in nonintegrable equations, analytical techniques typically rely on perturbative methods, asymptotic analysis, and/or variational approximations[77,113] An important example of a nonintegrable system with exact solutions for isolated solitary waves are the coupled mode equations for fiber Bragg gratings in optics.

Research on solitary waves and solitons remains one of the most vibrant areas of mathematics and physics [113]. It has had a broad and far-reaching impact in myriad fields ranging from the purest mathematics to experimental science. This has led to crucial results in integrable systems, nonlinear dynamics, optics, biophysics, supersymmetry, and more.

First, we construct soliton solutions to (1).

2.2 EXPLICIT CONSTRUCTION OF THE KdV SOLITON

It is illustrative to demonstrate the construction of the soliton solution of the KdV equation (1) explicitly. We start with the ansatz

$$\phi = \psi(y), \quad y = x - Ut \tag{2}$$

which describes a wave translating with speed U Inserting this into (1)

$$-U\psi' + \psi''' + 6\psi\psi' = 0 \tag{3}$$

where $' \equiv \frac{d}{dy}$. Integrating (3) and then multiplying the resulting equation by ψ' and integrating again yields

$$-\frac{U}{2}\psi^2 + \frac{1}{2}\left(\psi'\right)^2 + \psi^3 + G_1\psi + G_2 = 0 \tag{4}$$

where G_1 and G_2 are constants of integration.

We want a solution in the form of a localized pulse, so we need ψ, ψ', and all higher derivatives to vanish as $y\to\pm\infty$. This implies that $G_1 = G_2$ =0. [If one keeps nonzero constants, one can instead derive extended waves in the form of elliptic functions [130]

This gives

$$--\frac{U}{2}\psi^2 + \frac{1}{2}\left(\psi'\right)^2 + \psi^3 = 0 \tag{5}$$

Solving (5) by separation of variables yields

$$\phi(x,t) = \frac{U}{2}\operatorname{sech}^2\left\{\sqrt{U}\ \ (x - Ut - x_0)\right\} \tag{6}$$

where x_0 is a constant. We depict the solution (6) in Figure 4.

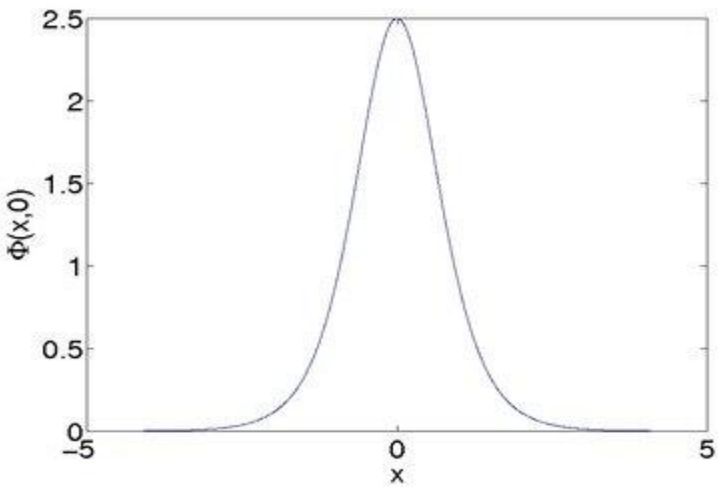

Figure 1: The wave $\phi(x, t)$ in equation (6) with U=5, $x(0) = 0$, and t=0 .

2.3 KDV SOLITON SOLUTION

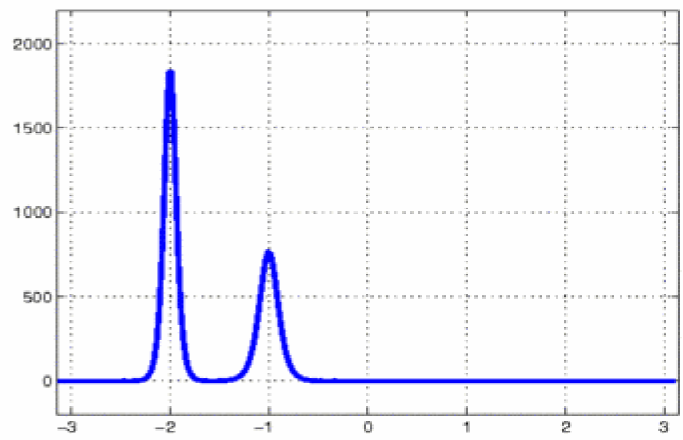

Figure 2: Collision between two soliton solutions of the KdV equation.

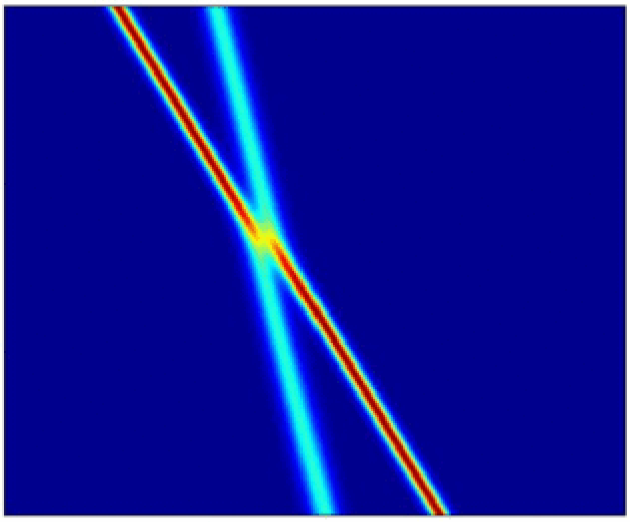

Figure 3: Space-time diagram of the collision in Figure 1.

Space is on the horizontal axis and time (increasing downward) is on the vertical axis. The heights of the waves are indicated by color, where bluer colors indicate smaller values and redder colors indicate larger values. Observe the phase shift that occurs when the two solitons collide.

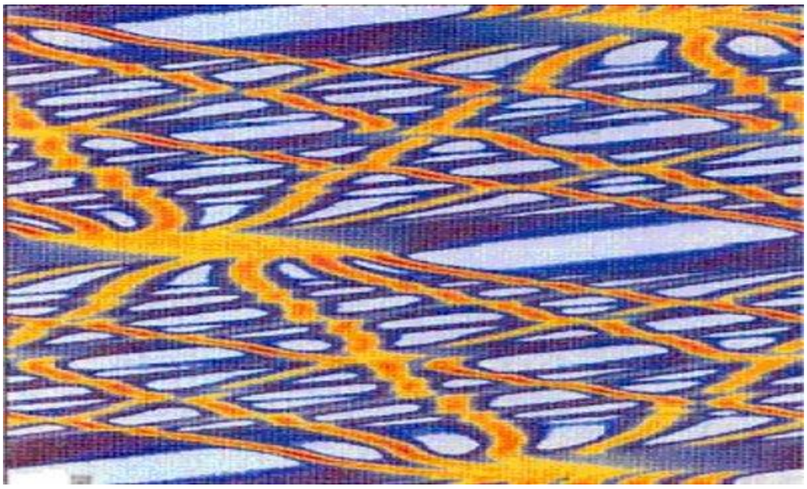

Figure 4: Near-recurrence in the Korteweg–de Vries (KdV) equation with a single-mode (sine wave) initial condition. This figure depicts a space-time diagram with a periodic spatial domain on the horizontal axis and time (increasing downward) on the vertical axis. The red–orange streaks indicate right-propagating, large-amplitude solitons that arise from the initial condition.

2.4 THE SOLITARY WAVE MENAGERIE

Since the discovery of solitary waves and solitons, a menagerie of localized pulses has been investigated in both one dimension and multiple spatial dimensions, though one must be nuanced when considering what constitutes a solitary wave (or even a localized solution) in multiple spatial dimensions. Many localized pulses have been given a moniker ending in "on" for conciseness, although they do not in general have similar interaction properties as solitons. The most prominent examples include the following:

2.4.1 ENVELOPE SOLITONS [113]

Solitary-wave descriptions of the envelopes of waves, such as those that arise from the propagation of modulated plane waves in a dispersive nonlinear medium with an amplitude-dependent dispersion relation. One typically uses the descriptor bright to describe solitary waves whose peak intensity is larger than the background (reflecting applications in optics) and the descriptor dark to describe solitary waves with lower intensity than the background.

2.4.2 SOLITARY WAVES WITH DISCONTINUOUS DERIVATIVES:

Examples of such solitary waves include compactons [107], which have finite (compact) support, and peakons, whose peaks have a discontinuous first derivative. There have also been studies of cuspons [113], which have a singularity in the first derivative rather than simply a discontinuity.

2.4.3 GAP SOLITONS [113, 32]

Solitary waves that occur in finite gaps in the spectrum of continuous systems. For example, gap solitons have been studied rather thoroughly in NLS equations with spatially periodic potentials and have been observed experimentally in the context of both nonlinear optics and Bose–Einstein condensation.

2.4.4 INTRINSIC LOCALIZED MODES (ILMS) [29 ,49]

ILMs, or discrete breathers, are extremely spatially-localized, time-periodic excitations in spatially extended, discrete, periodic (or quasiperiodic) systems. (At present, it is not clear whether analogous time-quasiperiodic solutions can be constructed for general lattice equations.) ILMs, which are localized in real space, arise in a large variety of nonlinear lattice models and are typically independent of the number of spatial dimensions of the lattice, the size of the lattice (which is, however, assumed to be large), and (for the most part) the precise choice of nonlinear forces acting on the lattice. The mechanism that permits the existence of ILMs has been understood theoretically for well over a decade, and such waves have now been observed in a large variety of physical systems. In common parlance, it is also typically assumed that intrinsic localized modes arise naturally from a system rather than due to impurities or defects. In this context, an ILM is a special type of discrete breather (which can be centered about an impurity and is otherwise as described above) rather than synonymous to a discrete breather, and many scholars also reserve the term "ILM" for modes that are stable or at least very long lived.

2.4.5 Q-BREATHERS [49]

Exact time-periodic solutions of spatially extended nonlinear systems that are continued from the normal modes of a corresponding linear system. In contrast to

ILMs, q-breathers are localized in normal-mode (Fourier) space, so that almost all of the energy is locked into a single Fourier mode for all time. (The label q refers to the wave number of the normal mode.) They also provide the best-known explanation for FPU recurrences.

2.4.6 TOPOLOGICAL SOLITONS [113]

Solitons, such as some solutions to the sine–Gordon equation, that emerges because of topological constraints. One example is a skyrmion, which is the solitary-wave solution of a nuclear model whose topological charge is the baryon number. Other examples include domain walls, which refer to interfaces that separate distinct regions of order and which form spontaneously when a discrete symmetry (such as time-reversal symmetry) is broken, screw dislocations in crystalline lattices, and the magnetic monopole. One-dimensional topological solitons are necessarily kinks, which we discuss below.

2.4.7 KINKS [113]

The only one-dimensional topological solitary wave, it represents a twist in the value of a solution and causes a transition from one value to another. Kinks can sometimes be represented using heteroclinic orbits, whereas pulse-like solitary waves can sometimes be represented using homoclinic orbits. Kinks are sometimes used to represent domain walls.

2.4.8 VORTEX SOLITONS[113]

A term often applied to phenomena such as vortex rings (a moving, rotating, toroidal object) and vortex lines (which are always tangent to the local vorticity). Coherent vortex-like structures also arise in dissipative systems.

2.4.9 DISSIPATIVE SOLITONS [113]

Stable localized structures that arise in spatially extended dissipative systems. They are often studied in the context of nonlinear reaction–diffusion systems.

2.4.10 OSCILLONS

A localized standing wave that arises in granular and other dissipative media that results from, e.g., the vertical vibration of a plate topped by a layer of free particles.

2.4.11 HIGHER-DIMENSIONAL SOLITARY WAVES [113]

Solitary waves and other localized (and partially localized) structures have also been studied in higher-dimensional settings. One example of a genuine two-dimensional soliton is the "lump" solution of the KP equation of the first type (i.e., the KP1 equation). This type of soliton decays algebraically rather than exponentially and is sometimes described as "weakly localized". The KP1 equation also has unstable line soliton solutions (a generalization of the soliton solutions of the KdV equation), which decay exponentially in all but a finite number of directions. The KP equation of the second type (i.e., the KP2 equation) differs from the KP1 equation in that it has the opposite sign in front of its diffusion term. The KP2 equation has stable line-soliton solutions, which (unlike line solitons in the KP1 equation) can merge with each other to form a single line soliton (which can, in turn, disintegrate into two separate line solitons).

Numerous generalizations of the above examples have also been investigated, as one can consider chains of solitons, discrete analogs of the above examples (such as discrete vortex solitons), semi-discrete examples (such as spatiotemporal solitary waves in arrays of optical fibers), one type of soliton "embedded" in another type, solitary waves in nonlocal media, quantum solitary waves, and more.

2.5. APPLICATIONS

Solitary waves of all flavors arise ubiquitously in fluid mechanics, optics, atomic physics, biophysics, and more [113]

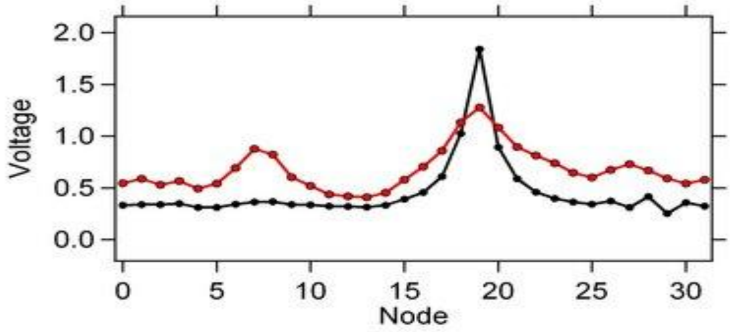

Figure 5

Experimental discrete breather, produced via modulational instability, in a bi-inductance electrical lattice.

It is impossible to discuss these manifestations exhaustively, so we show a few exciting figures and restrict ourselves to brief discussions of some of our favorite examples:

2.5.1 NONLINEAR OPTICS [113,10]

Solitary waves are omnipresent in nonlinear optics. There have been extensive experimental and theoretical investigations about both spatial solitary waves, in which nonlinearity balances diffraction, and temporal solitary waves, in which nonlinearity balances dispersion. From a mathematical perspective, continuous nonlinear Schrödinger (NLS) equations are among the hallmark models in nonlinear optics, as they describe dispersive envelope waves (via solitary-wave solutions of the NLS) of the electric field in optical bers, and discrete NLS (DNLS) equations can be used to describe the dynamics of pulses in, e.g., optical waveguide arrays and photorefractive crystals. Classes of solitary waves known as second-harmonic generation (SHG) solitary waves, which are so-named because they occur in (second-order nonlinearity) materials in optics, have been created experimentally in both spatial and temporal domains. Such materials have also been used to provide perhaps the only experimental generation of spatiotemporal solitary waves, in which there is a simultaneous balance of diffraction by self-focusing modulation and dispersion by phase modulation.

There have also been numerous studies of light bullets, which are three-dimensional localized pulses in self-focusing media with anomalous group dispersion. The properties of optical solitary waves can be manipulated experimentally through both "dispersion management" and "nonlinearity management"

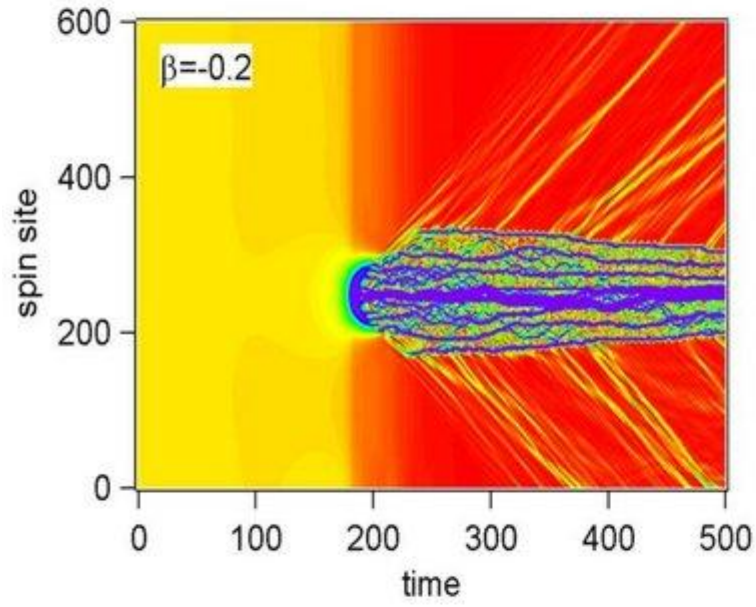

Figure 6

Simulation of spin-wave localization in an antiferromagnet with a demagnetization field. The areas with higher energy are shaded in bluer colors. The sample-shape parameter measures the ratio of demagnetization energy to exchange energy. Initially, one observes the formation of a single broad ILM in the entire 1024-spin lattice. Energy is then rapidly transferred to a smaller region from the rest of the lattice, so that the ensuing excitation breaks up into several virtually stationary and strongly localized defects.

2.5.2 BOSE–EINSTEIN CONDENSATES (BECS) [32]

At very low temperatures, particles in a dilute Bose gas can occupy the same quantum (ground) state, forming a BEC, a coherent cloud of atoms which appears as a sharp peak in both position and momentum space. As the gas is cooled, a large fraction of the atoms in the gas condense via a quantum phase transition, which occurs when the wavelengths of individual atoms overlap and behave identically. The macroscopic dynamics of BECs near zero temperature is modeled by an NLS equation known as the Gross–Pitaevskii (GP) equation. BEC solitary waves of numerous types have also been modeled using other models, such as DNLS equations. Because of the similarity of the model equations, many of the solitary-wave phenomena that were originally studied in the context of nonlinear optics arise here as well, and the extreme tunability of BECs has been a major boon for both theoretical and experimental studies. For example, bright solitary waves were created in Li atoms and gap solitons have been created in Rb. Additionally, there have been several theoretical studies on manipulating the properties of solitary waves in BECs via nonlinearity management (which can be achieved in principle by exploiting the properties of Feshbach resonances). Many novel types of solitary-wave structures have now been created in BEC laboratories, and research on nonlinear waves in BECs continues to develop at a rapid pace. One of the most important current experimental challenges for work on solitary waves in BECs (and also nonlinear optics) is the creation of stable two-dimensional and three-dimensional solitary waves in the presence of cubic self-focusing nonlinearity, as such structures must be stabilized in order to prevent them from collapsing (in accord with theoretical predictions).

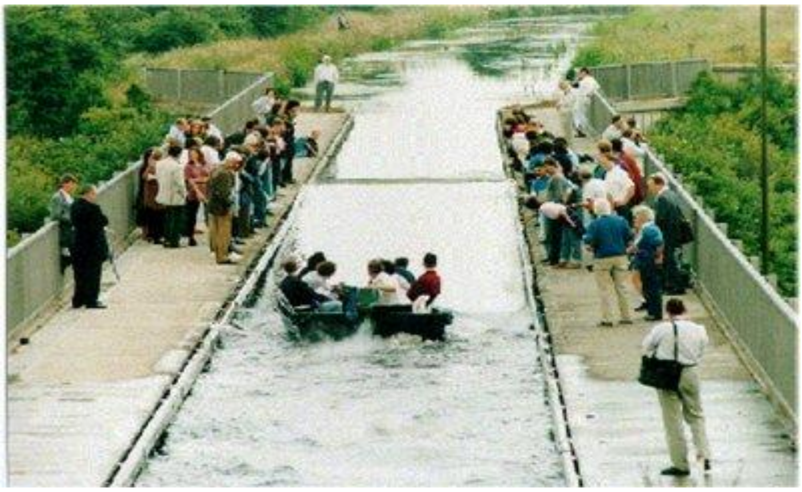

Figure 7

1995 recreation of Russell's soliton in the Union Canal. Figure courtesy of Chris Eilbeck and Heriot–Watt University.

2.5.3 WATER WAVES [113]

Russell's "wave of translation" was a water-wave soliton, and (as discussed above) Korteweg and de Vries derived their nonlinear wave equation to describe the shallow water waves that Russell had observed. The KdV equation arises in the long-wavelength limit, and shallow-water solitary waves have been the subject of numerous laboratory experiments. Solitary waves also arise in deep water, as shown by the pioneering work of Vladimir Zakharov who derived an envelope wave description whose limiting case satisfies an NLS equation. Additionally, solitary-wave solutions have been constructed in more sophisticated models in fluid dynamics, and there has been a lot of work on myriad types of solitary waves. For instance, various scientists have attempted to explain the large and seemingly spontaneous freak waves (or rogue waves) as solitary waves. Additionally, tidal bores have been explained in terms of dispersive shock waves, which consist (in spatial profile) of a leading pattern in the form of a solitary traveling wave and a trailing pattern in the form a wave train with slowly modulated amplitude that eventually asymptotes to a stationary state. Other interesting studies have focused on turbulent velocity fields that can arise from the breaking of solitary waves, 3D vortex structures under breaking waves, and the spilling and plunging of waves.

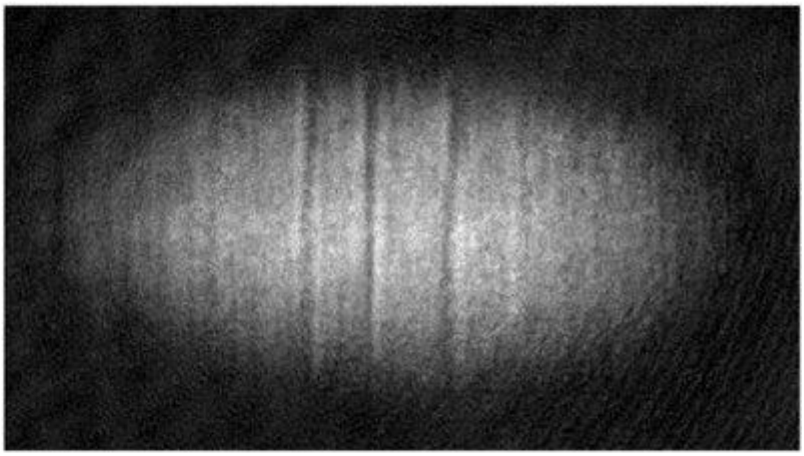

Figure 8

Three dark solitary waves in an Rb Bose–Einstein condensate (BEC). The intensity increases from dark to light, so these solitary waves are lower-density pulses in a higher-density background. The axial (horizontal) length of the BEC is about 250 microns.

2.5.4 BIOPHYSICS [130, 29]

There have been some attempts to use solitary-wave descriptions to describe various biophysical phenomena. One example is the Davydov soliton, which satisfies an equation that was designed to model energy transfer in hydrogen-bonded spines that stabilize protein α-helices The Davydov soliton represents a state composed of an excitation of amide-I and its associated hydrogen-bond distortion. It has been used to describe a local conformational change of the DNA α+helices and there now exists experimental evidence of such states. Another type of DNA solitary wave was introduced by Peyrard and Bishop, who interpreted solitary-wave solutions of a model for DNA denaturization as bubbles that appear in the DNA structure as temperature is increased. The Peyrard–Bishop model also admits ILM solutions, and ILMs have also been investigated both theoretically and experimentally in the context of biopolymers. Using a model similar to Davydov's, local modes in molecular crystals have also been described using solitary waves. More controversially, solitary waves have recently been used in neuroscience as an alternative to the accepted Hodgkin–Huxley model to describe the traveling of signals along a cell's membrane.

2.5.5 MICROELECTROMECHANICAL (MEM) AND NANOELECTROMECHANICAL (NEM) DEVICES [29]

Among the primary classes of systems in which ILMs have been studied are MEMs or NEMs consisting of arrays of nonlinear oscillators (such as cantilevers).

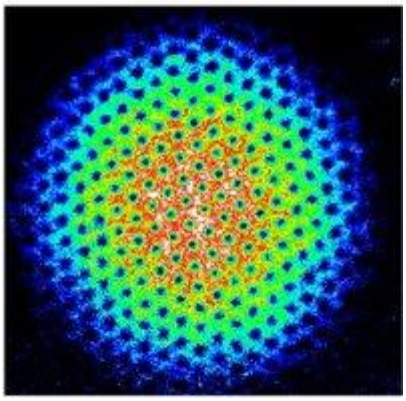

Figure 9

A vortex lattice in a rapidly rotating BEC. In this image, which is taken after the condensate expands when the trap is turned off; the diameter is about 900 microns. The trap has a diameter of about 125 microns, as the in-trap spacing between vortices is about 7 microns and there are about 18 rows of vortices across the BEC. Figure courtesy of Eric Cornell and Peter Engels.

2.5.6 JOSEPHSON JUNCTIONS [113, 29]

A Josephson junction is a nonlinear oscillator consisting of two weakly coupled superconductors that are connected by a non-conducting barrier. Such junctions might prove to be important for producing quantum-mechanical circuits such as superconducting quantum interference devices (SQUIDs). Additionally, some of the most visually striking ILMs have been observed in arrays of Josephson junctions. The first experimental realization of an array of such junctions revealed excitations that arose from spatially localized voltage drops at particular junctions as a homogeneous DC bias current traversed an annular array. Solitary waves in "long Josephson junctions", which are much longer than the intrinsic length scale known as the Josephson penetration depth (which is of the order 1–1000 μm), are known asfluxons because they contain one quantum of magnetic flux.

2.5.7 GRANULAR CRYSTALS

Granular crystals consist of a tightly-packed array of solid particles that deform when they contact each other. They are modeled by an FPU-like set of equations with an asymmetric potential (there is only a force when the particles are squeezing each other) that arises from the Hertzian description for contact between elastic particles. Granular crystals exhibit a highly nonlinear dynamic response, and the equations of motion give zero when they are linearized (although additional linear forces, such as gravity and precompression, can also be included with appropriate experimental setups). Taking a long-wavelength asymptotic limit of the equations of motion gives a partial differential equation whose only diffusion term is nonlinear. This equation admits traveling compacton solutions that match well with waves that have observed experimentally. (Such compactons give the best approximation that one can achieve with a continuum approximation, but a more accurate analysis reveals instead that the tails of the associated wave solutions of the original equations of motion exhibit a doubly-exponential decay.) Other types of solitary waves, including ILMs, have been observed in the presence of precompression.

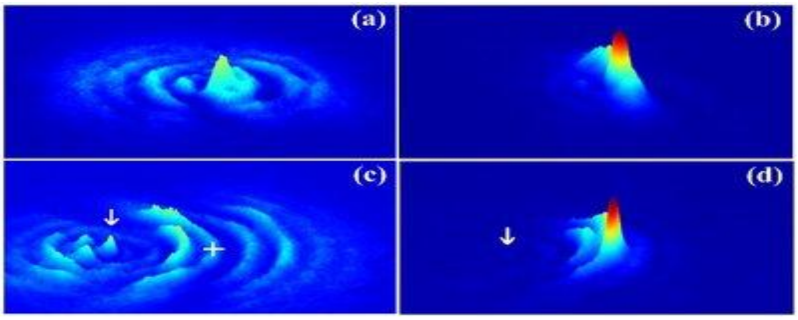

Figure 10

Transition from discrete diffraction [sub-panels (a, c)] to nonlinear self-trapping [sub-panels (b,d)] of a probe beam in a ring-shaped photonic lattice. The arrow indicates the center of the lattice and the cross indicates the input position of the probe beam.

2.5.8 SURFACE WAVES [113]

Numerous interesting nonlinear wave phenomena can occur on the surface of a "continuum" (e.g., fluids, solids, and appropriate granular materials—which can often be modeled using continuum descriptions), and some of them admit solitary-wave descriptions. Although it can be applied more broadly, the termsurface wave is often used to refer to a relatively specific class of examples. These include the pattern-forming standing waves called Faraday wavesthat form, e.g., on the surface of continua housed in vertically vibrated receptacles [similar phenomena have now also been seen in other settings, such as BECs , soliton-like oscillons that switch between peaks and craters and have been demonstrated in vertically-vibrated plates of granular materials, viscous fluids, and colloids; andacoustic surface waves, which travel along the surfaces of solid materials.

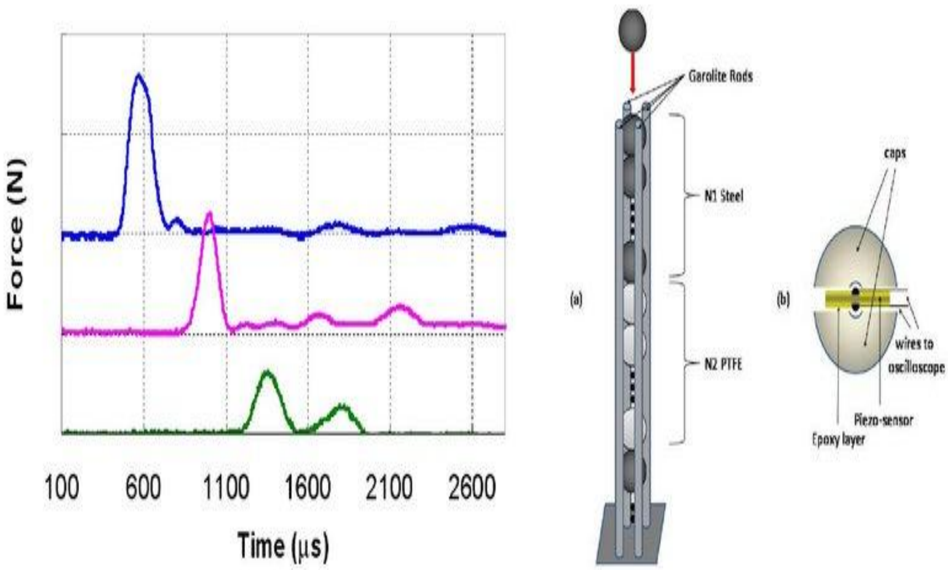

Figure 11

(Left) Compacton-like solitary waves in a granular chain consisting of a sequence of steel–Teflon dimers. (Each dimer consists of a spherical steel particle followed by a spherical Teflon particle.) The vertical axis shows the force in Newtons (horizontal lines are 2 N apart), and the horizontal axis shows the time in microseconds. The 13th and 33rd particles are made of steel, and the 24th particle is made of Teflon. (Right) Panel (a) shows the experimental configuration for chains composed of dimers consisting of N_1 steel particles and N_2 Teflon particles. Panel (b) shows the embedding of a Piezo-sensor to record the force in a particle. Both panels are from .

2.5.9 PLASMAS [113]

One of the convenient testbeds to study the dynamics of solitary waves has been plasmas, which consist of a large number of charged particles. For example, the KdV equation has been used to describe the local ion density (reflecting the local departure of the charge from neutrality) in a perturbation of the charge density. Other equations that admit soliton and solitary-wave solutions, including the Kadomtsev–Petviashvili (KP) equations and more complicated variants of both the KdV and KP equations, are also prominent in the study of plasmas. Dusty plasmas, which contain small suspended particles, have been modeled using nonlinear oscillator chains that admit several types of solitary-wave solutions (such as ILMs).

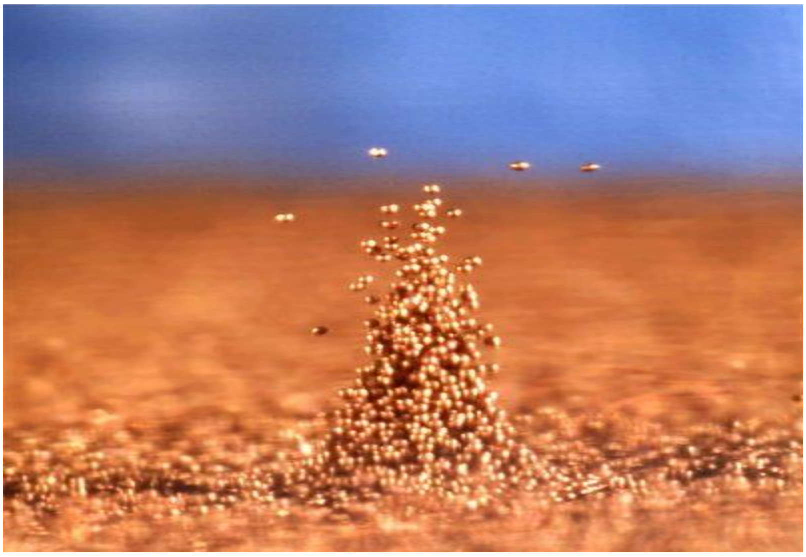

Figure 12

An oscillon in a vertically-vibrated layer of bronze beads.

2.5.10 FIELD THEORY

Solitons and their relatives, such as instantons, are also important in both classical and quantum field theory. Topological solitons such as monopoles, kinks, vortices, and skyrmions are key to the modern understanding of field theory. (Non-topological solitons such as Q-balls have generally played a less central role than their topological counterparts.) In (1+1)-dimensional quantum field theory, topological soliton solutions of the sine–Gordon equation can be mapped to elementary excitations of the Thirring model (an exactly solvable quantum field theory). This provides a toy model for more physically relevant examples in which the role of solitons is played by magnetic monopoles that can be mapped to electrically charged elementary particles via an equivalence that is given the name strong–weak duality or, more commonly, S-duality. S-duality is also an essential feature of string theory. Instantons give non-perturbative corrections to path integrals, and they play a crucial role in quantum field theory (especially in tunneling phenomena). Because of their algebraic structure, topological instantons can sometimes be constructed explicitly using methods from subjects such as twistor theory. Topological solitons also arise in various parts of string theory and super gravity (such as in studies of D-branes and NS-branes), as well as in the study of defects such as domain walls and cosmic strings.

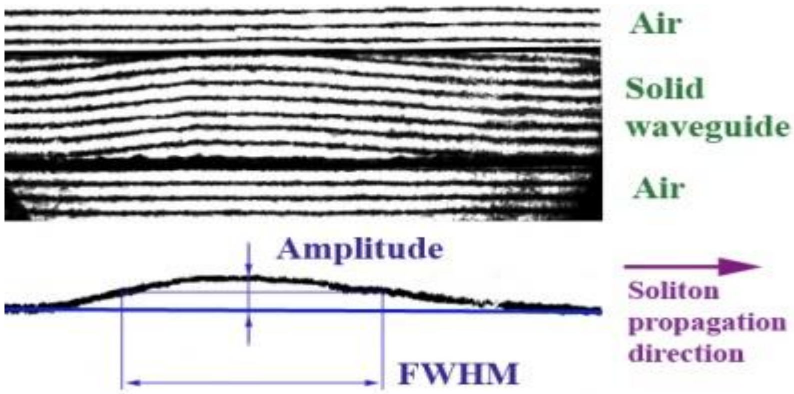

Figure 13

A holographic image of a bulk strain soliton in a Poly (methyl methacrylate) [PMMA] bar.

2.6 NONLINEAR SYSTEM

In mathematics, a nonlinear system [38] is one that does not satisfy the superposition principle, or one whose output is not directly proportional to its input; however, a linear system fulfills these conditions. In other words, a nonlinear system is any problem where the equation(s) to be solved cannot be written as a linear combination of the unknown variables or functions that appear in it (them). It does not matter if nonlinear known functions appear in the equations; in particular, a differential equation is linear if it is linear in the unknown function and its derivatives, even if nonlinear known functions appear as coefficients.

Nonlinear problems are of interest to engineers, physicists and mathematicians because most physical systems are inherently nonlinear in nature. Nonlinear equations are difficult to solve and give rise to interesting phenomena such as chaos.[128] Some aspects of the weather (although not the climate) are seen to be chaotic, where simple changes and in some cases even infinitesimal changes in one part of the system produce complex effects throughout. A system is said to be linear if it satisfy the property of homogenity and additivity.

2.7 EXAMLPES OF LINEAR SYSTEMS [65]

1. Wave propagation such as sound and electromagnetic waves
2. Electrical circuits composed of resistors, capacitors, and inductors
3. Electronic circuits, such as amplifiers and filters
4. Mechanical motion from the interaction of masses, springs, and dashpots (dampeners)
5. Systems described by differential equations such as resistor-capacitor-inductor networks.
6. The unity system where the output is always equal to the input
7. The null system where the output is always equal to the zero, regardless of the input

8. Differentiation and integration, and the analogous operations of first difference and running sumfor discrete signals

9. Small perturbations in an otherwise nonlinear system, for instance, a small signal beingamplified by a properly biased transistor

10. Convolution, a mathematical operation where each value in the output is expressed as thesum of values in the input multiplied by a set of weighing coefficients.

11.Recursion, a technique similar to convolution, except previously calculated values in the output are used in addition to values from the input

2.8 EXAMLPES OF NONLINEAR SYSTEMS [38,65]

1. Systems that do not have sinusoidal fidelity, such as electronics circuits for: peak detection, squaring, sine wave to square wave conversion, frequency doubling, etc.

2.Common electronic distortion, such as clipping, crossover distortion and slewing.

3. Multiplication of one signal by another signal, such as in amplitude modulation and automatic gain controls.

4. Hysteresis phenomena, such as magnetic flux density versus magnetic intensity in iron, or mechanical stress versus strain in vulcanized rubber.

5.Saturation, such as electronic amplifiers and transformers driven too hard

6.Systems with a threshold, for example, digital logic gates, or seismic vibrations that are strong enough to pulverize the intervening rock.

Formally, linear systems are defined by the properties of homogeneity, additivity, and shift invariance. Informally, most scientists and engineers think of linear systems in terms of static linearity and sinusoidal fidelity.

2.9 NONLINEAR DIFFERENTIAL EQUATIONS

A system of differential equations is said to be nonlinear if it is not a linear system Problems involving nonlinear differential equations are extremely diverse, and methods of solution or analysis are problem dependent. Examples of nonlinear differential equations are the Navier–Stokes equations in fluid dynamics, KdV equation, Sine-Gordon Equation, Nonlinear Schrodinger Equation, Volterra equations in biology etc.

One of the greatest difficulties of nonlinear problems is that it is not generally possible to combine known solutions into new solutions. In linear problems, for example, a family of linearly independent solutions can be used to construct general solutions through the superposition principle. A good example of this is one-dimensional heat transport with Dirichlet boundary conditions, the solution of which can be written as a time-dependent linear combination of sinusoids of differing frequencies; this makes solutions very flexible. It is often possible to find several very specific solutions to nonlinear equations, however the lack of a superposition principle prevents the construction of new solutions. They are difficult to study: there are almost no general techniques that work for all such equations, and usually each individual equation has to be studied as a separate problem.

2.10 INTEGRABLE SYSTEMS

PDEs that arise from integrable systems are often the easiest to study, and can sometimes be completely solved. A well-known example is the Korteweg–de Vries equation.

2.11 LIST OF SOME WELL-KNOWN CLASSICAL INTEGRABLE SYSTEMS

- Classical mechanical systems (finite-dimensional phase space)
- Harmonicoscillators in *n* dimensions
- Centralforce motion
- Two center Newtonian gravitational motion
- Geodesic motion on ellipsoids
- Neumann oscillator
- Lagrange, Euler and Kovalevskaya tops
- Integrable Clebsch and Steklov systems in fluids
- Calogero–Moser–Sutherland models

- Swinging Atwood's Machine with certain choices of parameters

2. Integrable lattice models

- Toda lattice
- Ablowitz–Ladik lattice
- Volterra lattice

3. Integrable systems of PDEs in 1 + 1 dimension

- Korteweg–de Vries equation
- Sine–Gordon equation
- Nonlinear Schrödinger equation
- Boussinesq equation (water waves)
- Nonlinear sigma models
- Classical Heisenberg ferromagnet model (spin chain)
- Classical Gaudin spin system (Garnier system)
- Landau–Lifshitz equation (continuous spin field)
- Benjamin–Ono equation
- Dym equation
- Three wave equation

4. Integrable PDEs in 2 + 1 dimensions

- Kadomtsev–Petviashvili equation
- Davey–Stewartson equation
- Ishimori equation

5. Other integrable systems of PDEs in higher dimensions

- Self-dual Yang–Mills equations

2.12 INVERSE SCATTERING TRANSFORM

In mathematics, [4] the **inverse scattering transform** is a method for solving some non-linear partial differential equations. It is one of the most important developments in mathematical physics in the past 40 years. The method is a non-linear analogue, and in some sense generalization, of the Fourier transform, which itself is applied to solve many linear partial differential equations.

The inverse scattering transform may be applied to many of the so-called exactly solvable models, that is to say completely integrable infinite dimensional systems. These include the Korteweg–de Vries equation, the nonlinear Schrödinger equation, the coupled nonlinear Schrödinger equations, the Sine-Gordon equation, the Kadomtsev–Petviashvili equation, the Toda lattice equation, the Ishimori equation, the Dym equation etc. A further, particularly interesting, family of examples is provided by the Bogomolny equations (for a given gauge group and oriented Riemannian 3-fold), the L^2 solutions of which are magnetic monopoles.

A characteristic of solutions obtained by the inverse scattering method is the existence of solitons, solutions resembling both particles and waves, which have no analogue for linear partial differential equations. The term "soliton" arises from non-linear optics.

The inverse scattering problem can be written as a Riemann–Hilbert factorization problem. This formulation can be generalized to differential operators of order greater than 2 and also to periodic potentials.

2.13 ZAKHAROV–SHABAT SYSTEM

The nonlinear Schrödinger equation is integrable: Zakharov and Shabat (1972) solved it with the inverse scattering transform. The corresponding linear system of equations is known as the Zakharov–Shabat system:

In 1972, Zakharov and Shabat [71] studied the Nonlinear Schrodinger equation, hereafter abbreviated the NLS equation: $iu_t + u_{xx} + 2|u|^2 u = 0$

This equation describes the stationary two-dimensional self-focusing and the associated transverse instability of a plane monochromatic wave. Unlike the linear Schrodinger equation, it contains a Soliton solution, thereby embodying the concept of a wave packet. It represents a balance between linear dispersion, which tends to break up the wave packet, and a focusing effect of the cubic nonlinearity, produced by self-interaction of the wave with itself. Zakharov and Shabat found a Lax pair for this equation and showed that one can solve it using the inverse scattering technique. This was indeed an important discovery, not only because it was a second nonlinear evolution equation solvable by this technique, but also because the associated linear eigenvalue problem that one has to consider was not the linear Schrodinger equation in this case.

2.14 AKNS

In 1973, Ablowitz, Kaup, Newell, and Segur [7,3] applied the Zakharov-Shabat inverse scattering formalism to the Sine-Gordon equation $u_{xt} = \sin(u)$ This equation describes the propagation of ultra-short optical pulses in resonantmedia, and also arises in statistical mechanics and condensed matter physics. In fact it had also been studied long ago in connection with the theory of surfaces of Constant negative curvature.