

CHAPTER 3

LATTICE DISCRETIZATION APPROACH TO NONLINEAR DIFFERENTIAL EQUATIONS

3.1 INTRODUCTION

Discretized nonlinear differential equations (KdV and nonlinear Schrodinger equation) have been solved by many authors by multi scale expansions. [127,134,18,22,75,118,126] In this paper we have solved nonlinear differential equations via lattice discretization. This is done by first removing the time dependence of the differential equation and then invoking lattice discretization. This results in a difference equation. For nonlinear equations, lattice discretization results in a nonlinear difference equation which must be solved via other methods. A matlab program has been written which solves the difference equation and plots the solutions from the recurrence relation itself. Via this technique we have solved Nonlinear Schrodinger equation, Toda Lattice equation of motion and Klein-Gordon equation. In each of these cases we have obtained Soliton solutions.

SOLITON is a self-reinforcing solitary wave (a wave packet or pulse) that maintains its shape while it travels at constant speed. Solitons are caused by a cancellation of nonlinear and dispersive effects in the medium.

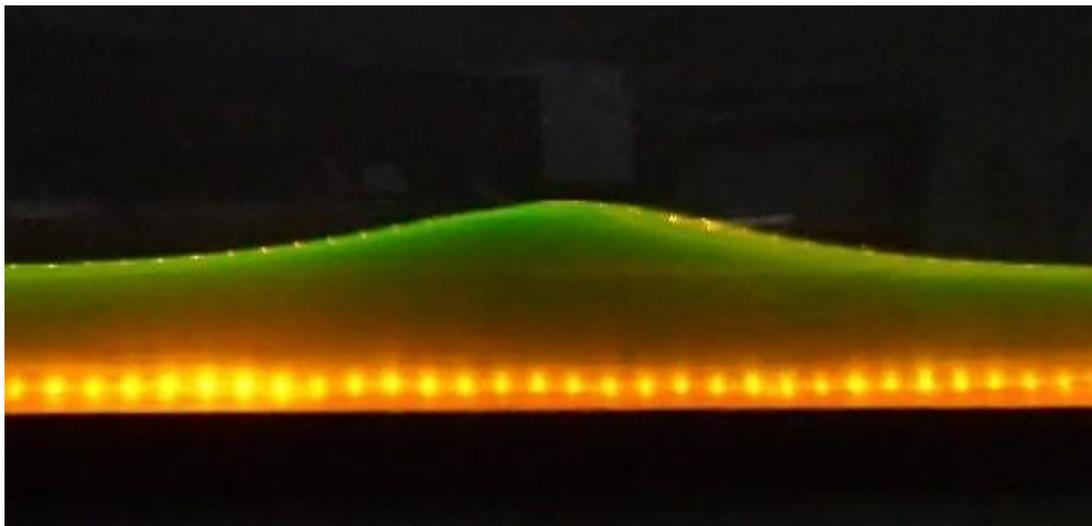


Figure 1 Solitary waves in a laboratory wave channel

3.2 GENERAL METHOD OF SOLUTION.

We have solved nonlinear differential equations via lattice discretization[31,63,88] This is done by first removing the time dependence of the differential equation and then invoking lattice discretization. This resulting difference equation is solved via Z transform using MATLAB. A drawback of this method is that some equations (Toda Lattice equation, Sine Gordon equation) have to be linearized before applying this technique. We first present this approach and the results obtained. The programs used are also given. The limitations of this approach are also discussed.

3.3 METHOD OF SOLUTION

The equation is first discretized. Discretized form of nonlinear differential equations has been studied by a number of authors [4,26,64,99,100,30]. We are essentially looking for Soliton solutions. Let the equation of motion be in terms of u_n . Now we look for periodic solutions of u_n . In general u_n is complex. Our main objective is to remove the temporal derivatives. This is done via the substitution

$$\psi(n,t) = e^{i\Lambda t} u(n) \quad (1)$$

With this substitution the temporal derivatives are removed. Note that when the system involves coupled equations (say for example $\psi(n,t)$ and $\phi(n,t)$) then (1) must be replaced by

$$\psi(n,t) = e^{i\Lambda_1 t} u(n) \quad (2)$$

$$\phi(n,t) = e^{i\Lambda_2 t} u(n) \quad (3)$$

Further all spatial derivatives are represented by their difference equation analogs. We now have a recursion equation. We solve this recursion equation using Z transform. The Z transformed equation is solved (i.e. arranged in partial fractions) and then we take the inverse Z transform. However what makes this approach very powerful is that we have used MATLAB's symbolic math routines to take the Z transform, the inverse Z transform and subsequently plot the solution. The MATLAB code and the plot for the various equations are also given.

3.4 NONLINEAR SCHRÖDINGER EQUATION (NLS)

The nonlinear Schrödinger equation is given by [39]

$$i \frac{\partial \psi}{\partial t} + \frac{\partial^2 \psi}{\partial x^2} + \psi |\psi|^2 = 0 \quad (4)$$

The discrete form of the above equation is

$$i \frac{\partial \psi(n)}{\partial t} + \frac{\partial^2 \psi(n)}{\partial x^2} + \psi(n) |\psi(n)|^2 = 0 \quad (5)$$

The system is defined on system of uniformly spaced points. Now we look for periodicity.

$$\psi(n,t) = e^{i\Lambda t} u(n) \quad (6)$$

Using (6) in (5) we get.

$$-\Lambda u(n) + c(u(n+1) - 2u(n) + u(n-1)) + u(n) |u(n)|^2 \quad (7)$$

Where u is a complex valued function, c is inverse of the square of the lattice spacing, n is nodal index

Using the matlab program given in Table 1 we obtain the plot output shown in Figure1. A characteristic feature of all nonlinear equations is extreme sensitivity to parameters. The output of the difference equation corresponding to NLS is plotted for different parameters.

3.5 MATLAB PROGRAM TO PLOT THE OUT PUT OF THE EQUATION

$$-\Lambda u(n) + c(u(n+1) - 2u(n) + u(n-1)) + u(n) |u(n)|^2$$

function znls

```
pn = sym('p(n)');
```

```
pn1 = sym('p(n+1)');
```

```
pn2 = sym('p(n+2)');
```

```
pnm1 = sym('p(n-1)');
```

```
syms n z c lambda
```

```
eq = 1.45*pn + 0.727*(pnm1 + pn1)
```

```
Zeq = ztrans(eq,n,z)
```

```
syms Pz
```

```

Zeq = subs(Zeq,{'ztrans(p(n),n,z)','p(0)','p(1)',},{Pz,1,2})
eq = collect(Zeq,Pz)
P = solve(eq,Pz)
p = iztrans(P,z,n);
p = simple(p)
pretty(p)
m = 1:100;
y = double(subs(p,n,m));
plot(m,y)
title('pn coefficient: 1.45 (pnm1 + pn1) coefficient 0.727')

```

Table 1

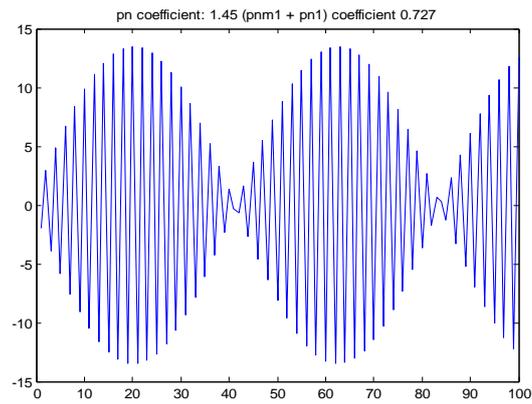


Figure 2 Soliton Profile for Nonlinear Schrodinger Equation.

3.6 TODA LATTICE:

The equation of motion of the Toda lattice is [39]

$$\ddot{\Phi}(n) = \exp(\Phi(n-1) - \Phi(n)) - \exp(\Phi(n) - \Phi(n+1)) \quad (8)$$

On expansion of the exponentials (neglecting higher order terms) to first order we get

$$\ddot{\Phi}(n) = \Phi(n-1) - 2\Phi(n) + \Phi(n+1) \quad (9)$$

Put $\Phi(n) = e^{i\Lambda t} \psi(n)$ (10)

using (10) in (9) we get

$$\Lambda^2 e^{i\Lambda t} \psi(n) = e^{i\Lambda t} \psi(n-1) - 2e^{i\Lambda t} \psi(n) + e^{i\Lambda t} \psi(n+1) \quad (11)$$

$$(2 - \Lambda^2) \psi(n) - \psi(n-1) - \psi(n+1) = 0 \quad (12)$$

Note that if $\Lambda = \pm\sqrt{2}$, $\psi(n+1) = -\psi(n-1)$

This shows successive deviation of the lattice points are equal and opposite. In other words the Lattice dimerizes. Examples of such dimerization can be found in the case of Poly-Acetylene. [119]

3.7 SOLUTIONS OF THE TODA LATTICE FOR VARIOUS PARAMETERS.

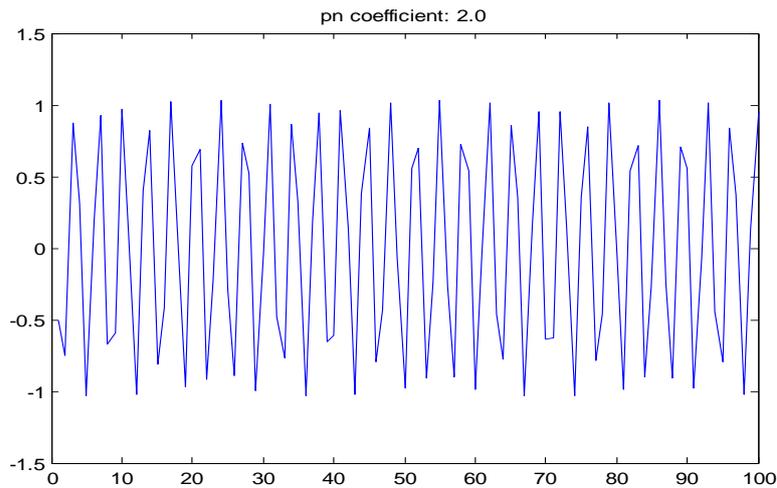


Figure 3 shows the profile of the Toda lattice Solitons.

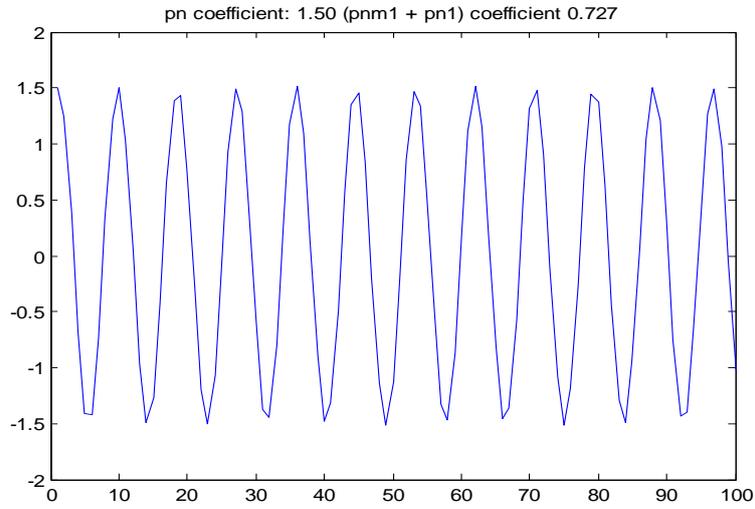


Figure 4 shows the profile of the Toda lattice Solitons.

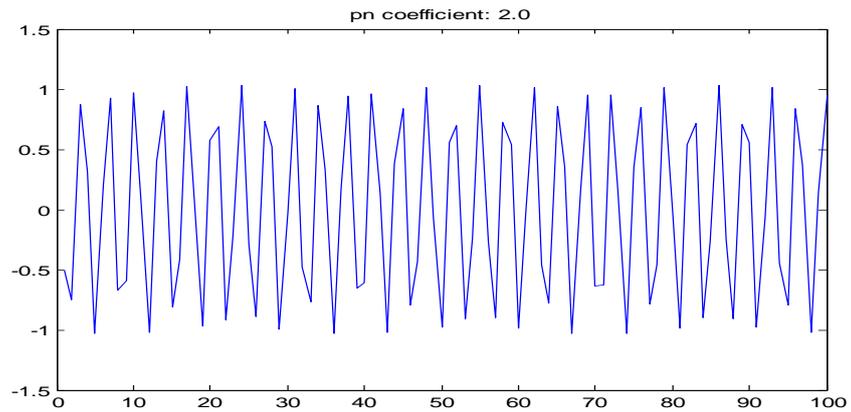


Figure 5 Toda lattice with different parameters.

3.8 KLEIN-GORDON EQUATION

The Klein Gordon equation is given by [39]:

$$\Phi_{xx} - \Phi_{tt} = m^2 \Phi \quad (13)$$

First we discretize the above equation as follows:

$$c^2 \Phi(n+1) - 2\Phi(n) + \Phi(n-1) - \Phi_{tt}(n) = m^2 \Phi(n) \quad (14)$$

$$+ \Phi(n) = e^{i\Lambda t} \psi(n) \quad (15)$$

Using (15) in (14) we obtain

$$\frac{\partial \Phi(n)}{\partial t} = i\Lambda e^{i\Lambda t} \psi(n) \quad (16)$$

$$\frac{\partial^2 \Phi(n)}{\partial t^2} = -\Lambda^2 e^{i\Lambda t} \psi(n) \quad (17)$$

Substituting (16) and (17) in (14) we get

$$c^2 e^{i\Lambda t} (\psi(n+1) - 2\psi(n) + \psi(n-1)) + \Lambda^2 e^{i\Lambda t} \psi(n) = m^2 e^{i\Lambda t} \psi(n) \quad (18)$$

$$c^2 (\Lambda^2 - m^2 - 2) \psi(n) - \psi(n+1) - \psi(n-1) = 0 \quad (19)$$

$\Lambda^2 - m^2 - 2 = 0$ means

$$\psi(n+1) = -\psi(n-1). \quad (20)$$

In other words neighboring lattice points have equal and opposite displacement. Note

that in this case $\Lambda = \pm (m^2 + 2)^{1/2}$.

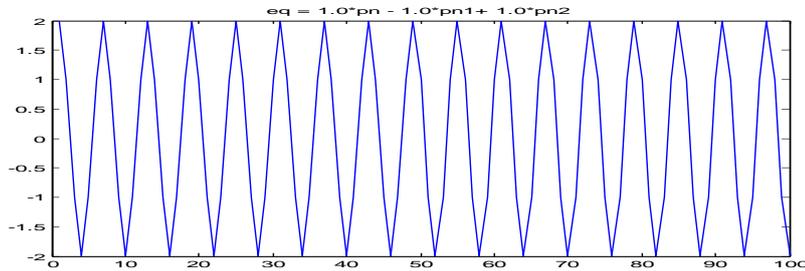


Figure 6 Profile of the Klein Gordon Solitons

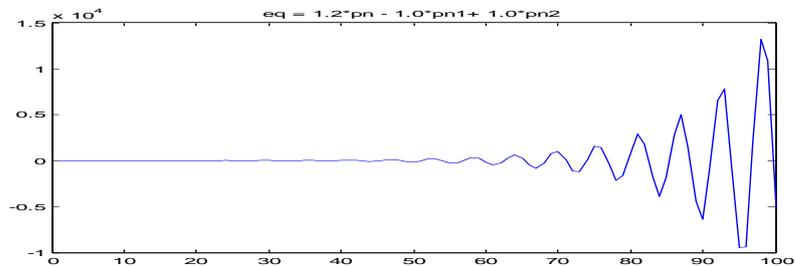


Figure 7 Klein Gordon Solitons with different parameters.

Note how the Klein Gordon Soliton solution gets perturbed by a slight change of parameters.

3.9 CONCLUSION

We have so far shown that the discretized form of the nonlinear equation of arbitrary complexity is easily solved via the technique outlined above. (However if the equation involves a function, for example $\sin(\varphi)$, in the Sine Gordon equation, $\sin(\varphi)$ must be linearized as $\sin(\varphi) \approx \varphi$. Also refer to the treatment of Toda lattice where the exponential had to be expanded to the first order to obtain the required format. This limitation can also be overcome by taking the Fourier transform or the Laplace Transform)

Note also that boundary conditions have not been incorporated in the above method of solution. Boundary conditions play an important role in the interference of Solitons (such as in Waves near a sea shore, Solitons in the sea and nonlinear crystals) The Z transformed equation expressed in terms of partial fractions allows one to understand the role of the nonlinearities. In all cases the degree of coarseness of the lattice ($1/(\Delta x)^2$) affects the results. Only at a critical range of coarseness of the lattice Soliton solutions occur. This result is in agreement with an earlier RG (Renormalization Group) study of Ising Lattices. It was found that as one approaches phase transition of the lattice the range of coupling between nearest neighbor spins (NN) increases (i.e. $NN \rightarrow NNN \rightarrow NNNN$ etc). Only in a specific range of coarseness is a phase transition caused, by the onset of Solitons (or Domain Walls), is observed. While the Soliton solutions themselves are different for the different equations of motion all equations show the same behavior in their approach to the Soliton solutions. This observation is again in agreement with RG studies of the Ising model. The symbolic math library of MATLAB gives the Z transform in terms of partial fractions. The Z transform converts the discretized into a recurrence relation. Recurrence relations have been mainly used for the study of probability based problem (e.g. Ehrenfest urn problem).