CHAPTER 5

LONG WAVE LENGTH SOLITON SOLUTION OF KdV EQUATION.

5.1 INTRODUCTION

In this chapter we will find the long wavelength soliton solution of KdV Equation. Recently H. Sakaguchi and B.A Malomed proposed a novel technique for finding the long wavelength solutions of the Gross Pitaevskii equation. We have extended their technique to KdV. In this case, in the long wavelength limit we find Soliton solutions using the above technique. We find bound state solutions for KdV equation.

One of the fascinating problems with large water bodies such as lake Ontario is that in the absence of wind one observes sinusoidal waves with a large time period of the order of days[58,50,95] (also known as long wavelength internal oscillations). However in the presence of wind one observes very large amplitude tanh type of waves. This phenomenon has not been explained so far. In this paper we attempt to set up a model to explain this phenomenon based on the KdV equation, in the long wave length limit and warm thermal currents in the lake .The warm thermal currents form a double well type of structure (see Figure. 1) which determines the nature of waves in the lake.

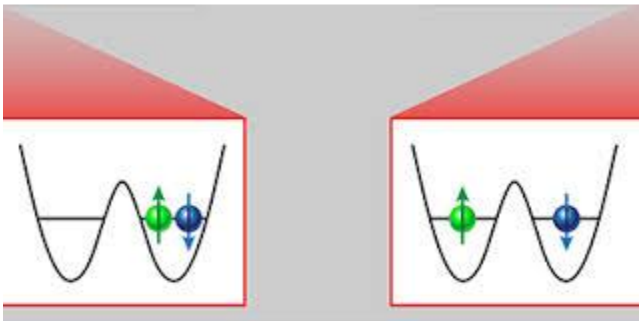

Figure 1

Traditionally KdV equation has been used to describe wave motion in shallow water bodies. Sakaguchi and B. A. Malomed in their seminal paper [111] proposed a novel expansion technique (2) to obtain the long wavelength Soliton solutions for the Gross-Pitaevskii equation. By substituting the expansion (2) in the KdV equation we obtain (by comparing coefficients) the effective equations in the long wavelength region. The effective equation is put in the form of a conservation law (4). The spatial component of the conservation law is the Schrodinger equation. In the limit where the nearest neighbor interactions are strong one obtains harmonic solutions. In the opposite limit one finds tanh soliton solutions.

5.2 KdV EQUATION

The KdV equation is

$$\psi_t + \psi_{xxx} - 6\psi\psi_x = 0 \tag{1}$$

In the long wavelength we look for solutions of the form [111]

$$\psi(x,t) = \psi^{(0)}(x,t) + \psi^{(1)}(x,t)\cos(2x) + ... \tag{2}$$

where $\psi^{(0)}(x,t)$ and $\psi^{(1)}(x,t)$ are slowly varying functions of x and t in comparison to cos(2x) and $\psi^{(1)}_{xx}(x,t) << \psi^{(0)}(x,t)$. This condition implies that $\psi^{(0)}(x,t) \approx \psi(x,t)$. Since $\psi(x,t)$ is a moving Soliton solution, this is only possible if $\psi^{(0)}(x,t)$ is also in the frame moving with the Soliton. Hence $\psi^{(1)}(x,t)$ describes the variation of the moving Soliton frame with respect to the stationary observer frame. The role of $\psi^{(1)}(x,t)$ is clarified below.

Substituting (2) in the KdV equation, and collecting the coefficients of cos(2x) we get

$$\psi^{(1)}_t + \psi^{(1)}_{xxx} - 12\psi^{(1)}_x - 6\psi^{(1)}\psi^{(0)}_x - 6\psi^{(0)}\psi^{(1)}_x + 12\psi^{(1)}\psi^{(1)}\sin 2x = 0 \tag{3}$$

Taking expansion around the point zero we get

$$\psi^{(1)}_t + \psi^{(1)}_{xxx} - 12\psi^{(1)}_x - 6\psi^{(1)}\psi^{(0)}_x - 6\psi^{(0)}\psi^{(1)}_x = 0 \tag{4}$$

This equation may be written as a conservation law as

$$\psi_t^{(1)} + \frac{d}{dx}\left[\psi_{xx}^{(1)} - 12\psi^{(1)} - 6\psi^{(1)}\psi^{(0)}\right] = 0 \tag{5}$$

In this asymptotic limit the spatial component of this conservation equation is the eigen value equation,

$$\psi_{xx}^{(1)} - V\psi^{(1)} = 0 \tag{6}$$

where $V = 6[2 + \psi^{(0)}]$ (7)

Note here that the effective potential in (6) is given in terms of $\psi^{(0)}(x,t)$, the solution of the KdV equation. Now the solution of the KdV equation [81] is given by

$$\psi^{(0)}(x,t) = \frac{1}{2}c\,\mathrm{sech}^2\left[\frac{\sqrt{c}}{2}(x - ct)\right] \tag{8}$$

which may be written as $\psi^{(0)}(x,t) = 2k^2 sech^2\left[k\psi^{(1)}\right]$ (9)

where $k = \frac{\sqrt{c}}{2}$ and $\psi^{(1)} = (x - ct)$ (10)

Assuming $\psi^{(1)}$ is the variable with respect to a frame of reference moving with a velocity c. $\psi^{(1)}$ is in reality the frame of reference of the Soliton moving with a velocity c. As mentioned earlier $\psi^{(1)}(x,t)$ describes the motion of the Soliton frame with respect to the observer frame. An important step in the solution of the KdV equation was provided by Gardner et al. [53], who proposed that it could be studied through the properties of the one-dimensional Schrödinger equation for potential V. Following Gardner (6) may be written as

$$\psi_{xx}^{(1)}(x,t) + (\lambda - V)\psi^{(1)} = 0 \tag{11}$$

and λ is the eigenvalue of the equivalent Schrodinger equation . Note here that the solution of the KdV equation ψ (or $\psi^{(0)}$) now acts as the potential in the equivalent Schrodinger equation. This is the signature of inverse scattering transform. For all nonlinear equations the solution of the nonlinear equation acts as a potential in the equivalent Schrodinger equation. Now

$$Sech(k\psi^{(1)}) = 1 - k^2\left(\psi^{(1)}\right)^2 + \frac{k^4\left(\psi^{(1)}\right)^4}{4}..... \tag{12}$$

where we assume $k\psi^{(1)} << 1$ and higher order terms have been neglected. We rewrite (11) by substituting (12) in (11) to obtain

$$\psi_{xx}^{(1)}(x,t) + \left(\lambda' - \left[-(k\psi^{(1)})^2 + \frac{\left(k\psi^{(1)}\right)^4}{4}\right]\right)\psi^{(1)} = 0 \tag{13}$$

$$\lambda' = \lambda - 13$$

The effective potential is therefore

$$V_{eff} = -(k\psi^{(1)})^2 + \frac{\left(k\psi^{(1)}\right)^4}{4} \tag{14}$$

Keeping (10) in mind we note that (14) represents a double well potential moving with a velocity c. This represents a double well with minima at $\pm\frac{1}{k\sqrt{3}} = \pm\frac{2}{\sqrt{3c}}$ where we have used the results of Krumhansl and Schrieffer [80] and $k = \frac{\sqrt{c}}{2}$ where c is the phase velocity. The depth of the potential well, as given by [80] is -1. Note that the depth of the potential well is constant. We note that since (14) corresponds to a double well one may write the effective Lagrangian for the above system as

$$L = \left(\frac{d\psi^{(1)}}{dt}\right)^2 + \frac{A}{2}\left(\psi^{(1)}\right)^2 + \frac{B}{4}\left(\psi^{(1)}\right)^4 + \frac{c_0}{2}\left(\psi^{(1)}\right)'' \tag{15}$$

where A is negative and B is positive. Here the primes indicate derivative with respect to x. The fourth term arises as result of elastic energy due to squeezing of nearest neighbor elements and c_0 is the elastic constant. The equation of motion in this effective potential may be computed using Euler Lagrange equation. One obtains

$$\frac{d^2\psi^{(1)}}{dt^2} + A\psi^{(1)} + B\left(\psi^{(1)}\right)^3 - c_0^2\left(\psi^{(1)}\right)'' = 0 \tag{16}$$

Using $\psi^{(1)} = f(x-ct)$ one obtains

$$(c^2 - c_0^2)f'' + Af + Bf^3 = 0 \tag{17}$$

To reduce the above to the dimensionless form [80] we use

$$\frac{\left(c_0^2 - v^2\right)}{|A|} = \xi^2, \frac{f}{\psi_0} = \eta, \frac{(x - vt)}{\xi} = s \tag{18}$$

we obtain
$$\frac{d^2\eta}{ds^2} + \eta + \eta^3 = 0 \tag{19}$$

In the limit $\eta^3 << 1$, (19) becomes
$$\frac{d^2\eta}{ds^2} + \eta = 0 \tag{20}$$

whose solution is
$$\psi = \psi_0 \sin\left(\frac{(x - vt)}{\xi}\right) \tag{21}$$

In the opposing limit $\eta^3 \approx 1$ the solution is

$$\psi = \psi_0 tanh\left(\frac{(x - vt)}{\xi\sqrt{2}}\right) \tag{22}$$

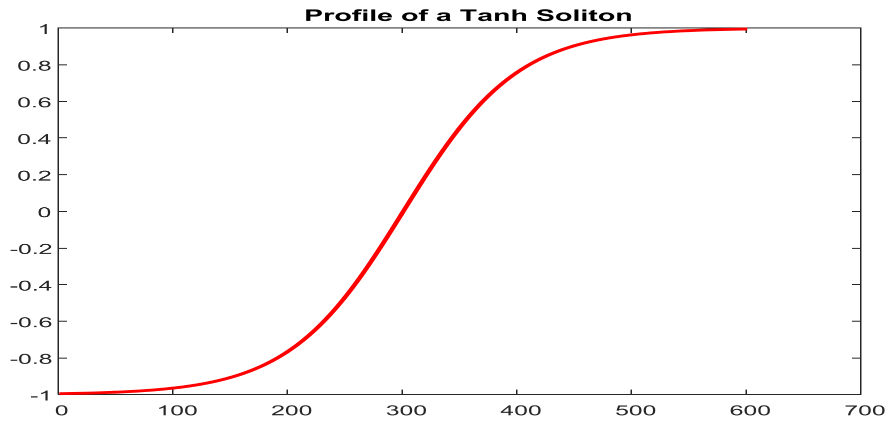

Figure 2

Here the x axis is a measure of the distance covered by the Soliton.The y axis is a measure of the amplitude of the Soliton. Note that Solitons can occur in the DNA lattice or in Oceans. KdV equation which describes wave motion in shallow water bodies admits tanh Soliton solutions in the long wavelength limit.

5.3 CONCLUSION

We have obtained both harmonic and tanh Soliton solutions of the KdV equation in the long wavelength limit. Both solutions have in fact been observed in large surface water lakes such as Lake Ontario [58, 50, 95].

A double well structure is created in lakes by the warm currents. Cold water from the surface of the lake shores sinks to the bottom while warm water rises to the surface producing the hump of the double well near about the centre of the lake. In the absence of wind one observes sinusoidal waves. Such waves have a time period of the order of days [58, 50]. However winds of sufficient strength are able to raise the energy of the waves above the hump of the double well. As a result in the presence of wind one observes large amplitude Tanh Soliton type of waves [95].