2 METHOD OF SUBSTITUTIO .

2.1 Change of variable.

Let $I = \int f(x)\,dx$, and let $x = \phi(z)$.

Then, by definition, $\frac{dI}{dx} = f(x)$ and $\frac{dx}{dz} = \phi'(z)$.

Now, $\frac{dI}{dz} = \frac{dI}{dx}\frac{dx}{dz} = f(x)\phi'(z) = f\{\phi(z)\}\phi'(z)$.

$\therefore$ by definition, $I = \int f\{\phi(z)\}\phi'(z)\,dz$.

Note 1. Thus, if in the integral $\int f(x)\,dx$ we put $x = \phi(z)$; we are to replace x by $\phi(z)$ in the expression $f(x)$ and also we are to replace dx by $\phi'(z)\,dz$ and then we have to proceed with the integration with z as the new variable. After evaluating the integral we are to replace z by the equivalent expression in x,

Note that though from $x = \phi(z)$ we can write $\frac{dx}{dz} = \phi'(z)$ in making our substitution in the given integral, we generally use it in the differential form $dx = \phi'(z)\,dz$. It really means that when x and z are connected by the relation $x = \phi(z)$, I being the function of x whose differential coefficient with respect to x is $f(x)$, it is, when expressed in terms of z, identical with the function whose differential coefficient with respect to z is $f\{\phi(z)\}\phi'(z)$ which later, by a proper choice of $\phi(z)$, may possibly be of a standard form, and therefore easy to find out.

Note 2. Sometimes it is found convenient to make the substitution in the form $\psi(x) = z$ where corresponding differential form will be $\psi'(x)\,dx = dz$; by means of these two relations, $f(x)\,dx$ is transformed into the form $F(z)\,dz$.

2.2 Illustrative Examples.

Ex. 1. *Integrate* $\int (a+bx)^n\,dx$.

Put $a + bx = z$. $\therefore$ $b\,dx = dz$. $\therefore$ $dx = (1/b)\,dz$.

$$\therefore\ I = \int z^n \frac{1}{b}dz = \frac{1}{b}\int z^n\,dz = \frac{1}{b}\frac{z^{n+1}}{n+1} = \frac{1}{(n+1)b}.(a+bx)^{n+1}.$$

Ex. 2. *Integrate* $\int \frac{dx}{x\sqrt{(x^2-a^2)}}$.

Put $x = a\sec\theta$. $\therefore$ $dx = a\sec\theta\tan\theta\, d\theta$.

$$\therefore\ I = \int \frac{a\sec\theta\tan\theta\, d\theta}{a\sec\theta . a\tan\theta} = \frac{1}{a}\int d\theta = \frac{1}{a}\theta = \frac{1}{a}\sec^{-1}\frac{x}{a}.$$

Cor. $\int \frac{dx}{x\sqrt{(x^2-1)}} = \sec^{-1} x$.

Ex. 3. *Integrate* $\frac{\sin^{-1} x}{\sqrt{(1-x^2)}}\, dx$.

put $\sin^{-1} = z$. $\therefore$ $\frac{1}{\sqrt{(1-x^2)}}\, dx = dz$.

$$\therefore\ I = \int z dz = \tfrac{1}{2} z^2 = \tfrac{1}{2}(\sin^{-1} x)^2.$$

Ex. 4. *Show that*

(i) $\int \tan x\, dx = \log|\sec x|$. (ii) $\int \cot x\, dx = \log|\sin x|$.

(i) Put $\cos x = z$; then $-\sin x\, dx = dz$.

$$\therefore\ I = \int \frac{\sin x}{\cos x}\, dx = -\int \frac{dz}{z} = -\log z.$$

$$= -\log\cos x = \log\frac{1}{\cos x} = \log|\sec x|$$

(ii) Similarly, by substituting $\sin x = z$, this result follows.

Otherwise:

(i) $\int \tan x\, dx = \int \frac{\sec x\tan x}{\sec x}\, dx = \log|\sec x|$.

(ii) $\int \cot x\, dx = \int \frac{\cos x}{\sin x}\, dx = \log|\sin x|$. [See Ex. 5 below]

Ex. 5. *Show that*

$$\int \frac{f'(x)}{f(x)}\, dx = \log|f(x)|.$$

Put $f(x) = z$. $\therefore f'(x)\,dx = dz$.

$\therefore\ I = \int \frac{dz}{z} = \log|z| = \log|f(x)|$.

Hence,

If the integrand be a fraction such that its numerator is the differential coefficient of the denominator, then the integral is equal to $\log|\text{denominator}|$.

Thus, $\int \frac{\cos x - \sin x}{\sin x + \cos x}\,dx = \log|(\sin x + \cos x)|$.

$$\int \frac{2ax+b}{ax^2+bx+c}\,dx = \log\left|(ax^2+bx+c)\right|.$$

The principle is also illustrated in Ex. 4 above.

Ex. 6. *Integrate* $\int \frac{2\sin x}{5+3\cos x}\,dx$.

I can be written as $-\frac{2}{3}\int \frac{-3\sin x}{5+3\cos x}\,dx$.

Now, since the numerator of the integrand is the differential coefficient of the denominator,

$\therefore\quad I = -\frac{2}{3}\log|(5+3\cos x)|$.

Ex. 7. *Integrate* $\int \frac{dx}{\sqrt{(x+a)}+\sqrt{(x+b)}}$.

Multiplying the numerator and denominator by $\sqrt{x+a}-\sqrt{x+b}$, we have

$$I = \int \frac{\sqrt{x+a}-\sqrt{x+b}}{a-b}\,dx = \frac{1}{a-b}\left[\int\sqrt{x+a}\,dx - \int\sqrt{x+b}\,dx\right].$$

.Putting $x + a = z$, so that $dx = z$.

$$\int\sqrt{x+a}\,dx = \int\sqrt{z}dz = \frac{2}{3}z^{3/2} = \frac{2}{3}(x+a)^{3/2}.$$

Similarly, $\int\sqrt{x+b}\,dx = \frac{2}{3}(x+b)^{3/2}$.

$\therefore\ I = \frac{2}{3}\frac{1}{a-b}[(x+a)^{3/2} - (x+b)^{3/2}]$.

Ex. 8. *Integrate* $\int \frac{(a+bx)^2}{(a'+b'x)^3}\,dx$.

Put $a'+b'x=z$, or, $x=\frac{z-a'}{b'}$. $\therefore\ dx=\frac{1}{b'}dz$.

Now the given integral becomes

$$\int \frac{\{a+\frac{b}{b'}(z-a')\}^2}{z^3}\frac{dz}{b'}=\frac{1}{b'^3}\int\frac{(bz+ab'-a'b)^2}{z^3}\,dz.$$

$$=\frac{1}{b'^3}[b^2\int\frac{dz}{z}+2b(ab'-a'b)\int\frac{dz}{z^2}+(ab'-a'b)^2\int\frac{dz}{z^3}].$$

$$=\frac{b^2}{b'^3}\log z-\frac{2b(ab'-a'b)}{b'^3}\frac{1}{z}-\frac{(ab'-a'b)}{2b'^3}\frac{1}{z^2}.$$

$$=\frac{b^2}{b'^3}\log(a'+b'x)-\frac{2b(ab'-a'b)}{b'^3(a'+b'x)}-\frac{(ab'-a'b)^2}{2b'^3(a'+b'x)^2}.$$

Note. By the same process we can integrate $\int\frac{(a+bx)^m}{(a'+b'x)^n}\,dx$, where m is a positive integer, n being a rational number. [Cf. §9.13]

Ex. 9. *Integrate* $\int\frac{dx}{x^3(a+bx)^2}$.

Put $a+bx=zx$, or, $\frac{a}{x}+b=z$. Then $-\frac{a}{x^2}dx=dz$.

The given integral then

$$=-\frac{1}{a}\int\frac{dz}{x.z^2x^2}=-\frac{1}{a}\int\frac{dz}{z^2}\left(\frac{z-b}{a}\right)^3.$$

$$=-\frac{1}{a^4}\int(z-3b+\frac{3b^2}{z}-\frac{b^3}{z^2})\,dz.$$

$$=-\frac{1}{a^4}[\frac{z^2}{2}-3bz+3b^2\log z+\frac{b^3}{z}].$$

$$= -\frac{1}{a^4}\left[\frac{1}{2}\left(\frac{a+bx}{x}\right)^2 - 3b\left(\frac{a+bx}{x}\right)\right.$$

$$\left. + 3b^2 \log\frac{a+bx}{x} + b^3\left(\frac{x}{a+bx}\right)\right].$$

Note. By the same substitution the integral $\int \frac{dx}{x^m (a+bx)^n}$ can be obtained where m and n are positive integers, or even when they are fractions such that $m + n$ is a positive integral greater than unity. For another method see §9.13.

EXAMPLES II(A)

Integrate the following :-

1. (i) $\int e^{\tan^{-1} x} \frac{1}{1+x^2}\, dx$. (ii) $\int e^{a \sin^{-1} x} \frac{1}{\sqrt{(1-x^2)}}\, dx$.

(iii) $\int \frac{\cos(\log x)}{x}\, dx$. (iv) $\int \frac{\cos^2 x}{\sin^4 x}\, dx$.

(v) $\int \frac{\cos 4x - \cos 2x}{\sin 4x - \sin 2x}\, dx$. (vi) $\int \frac{dx}{\operatorname{cosec} 2x - \cot 2x}$.

2. (i) $\int x\sqrt{x^2+1}\; dx$. (ii) $\int x^2\sqrt{a^3+x^3}\; dx$.

3. (i) $\int \frac{1}{\sqrt{x}} \cos\sqrt{x}\, dx$. (ii) $\int \frac{dx}{1+\cos x}$.

4. (i) $\int \frac{\sqrt{(\tan x)}}{\sin x \cos x}\, dx$. (ii) $\int \sqrt{\frac{\sin x}{\cos^5 x}}\, dx$.

5. (i) $\int \frac{1+\cos x}{\sqrt[3]{(x+\sin x)}}\, dx$. (ii) $\int \frac{1+\cos x}{x+\sin x}\, dx$.

6. (i) $\int \frac{\tan(\log x)}{x}\, dx$. (ii) $\int \frac{dx}{x\log x}$.

7. (i) $\int \frac{\cos x\, dx}{\sqrt{(1+\sin x)}}$. (ii) $\int \frac{\cos x\, dx}{(a+b\sin x)^2}$.

8. (i) $\int \frac{dx}{x^2 \sqrt{(1-x^2)}}$.

[Put $x = \sin\theta$.]

(ii) $\int \frac{dx}{x^2 \sqrt{(1+x^2)}}$.

(iii) $\int \frac{dx}{(1-x^2)\sqrt{(1-x^2)}}$.

(iv) $\int \frac{dx}{(1+x^2)\sqrt{(1+x^2)}}$.

9. (i) $\int \frac{e^x - 1}{e^x + 1} dx$.

(ii) $\int \frac{dx}{e^x + 1}$.

[Multiply the numerator and denomitor of (i) by $e^{-x/2}$, and that of (ii) by e^{-x}.]

10. (i) $\int \frac{\tan x}{\log \cos x} dx$.

(ii) $\int \frac{\cot x}{\log \sin x} dx$.

(iii) $\int \frac{\sec x \operatorname{cosec} x}{\log \tan x} dx$.

(iv) $\int \frac{\sec x \, dx}{\log(\sec x + \tan x)}$.

11. (i) $\int \frac{\sin 2x \, dx}{a \sin^2 x + b \cos^2 x}$.

(ii) $\int \frac{\tan x \, dx}{a + b \tan^2 x}$.

(iii) $\int \frac{\sin 2x \, dx}{(a^2 \cos^2 x + b^2 \sin^2 x)^2}$.

(iv) $\int \frac{\tan x \sec^2 x}{(a^2 + b^2 \tan^2 x)^2} dx$.

12. (i) $\int x \sin x^2 dx$.

(ii) $\int \frac{dx}{\sin x \cos x}$.

13. (i) $\int \frac{3x - 1}{\sqrt{(3x^2 - 2x + 7)}} dx$.

(ii) $\int \frac{x dx}{\sqrt{(x^2 - a^2)}}$.

14. (i) $\int \frac{dx}{(1 + x^2)\sqrt{(\tan^{-1} x + 3)}}$.

(ii) $\int \frac{\sec^4 x \, dx}{\sqrt{(\tan x)}}$.

15. (i) $\int \frac{e^{2x}}{e^x + 1} dx$.

(ii) $\int \frac{dx}{(e^x - 1)^2}$.

(iii) $\int \frac{dx}{\sqrt{(e^x - 1)}}$.

16. $\int \frac{e^x (1+x)}{\cos^2 (xe^x)}\, dx$. [Put $xe^x = z$.]

17. (i) $\int \frac{dx}{\sqrt{(x+1)} - \sqrt{(x+1)}}$. (ii) $\int \frac{dx}{\sqrt{(2x+5)} + \sqrt{(2x-3)}}$

(iii) $\int \frac{dx}{\{(x-3)+(x-4)\}\sqrt{\{(x-3)(x-4)\}}}$.

18. (i) $\int \frac{dx}{\sqrt{x} + x}$. [Put $\sqrt{x} = z$] (ii) $\int \frac{x\,dx}{(2x+1)^3}$.

19. (i) $\int \frac{dx}{\sqrt{x-1}}$. (ii) $\int \frac{x}{\sqrt{x+1}}\, dx$.

20. (i) $\int (3x+2)\sqrt{2x+1}\, dx$. (ii) $\int x^3 \sqrt{(x+a)}\, dx$.

21. (i) $\int \frac{1+x}{1-x}\, dx$. (ii) $\int \frac{x^6}{x-1}\, dx$.

22. (i) $\int \frac{2x+3}{3x+4}\, dx$. (ii) $\int \frac{x}{a+bx}\, dx$.

23. (i) $\int \frac{2x+1}{\sqrt{(3x+2)}}\, dx$. (ii) $\int \frac{x}{\sqrt[3]{(a+bx)}}\, dx$.

24. $\int \sqrt{\frac{a+x}{a-x}}\, dx$. [Put $x = a\cos 2\theta$] **25.** $\int \frac{2x^3+3x^2+4x+5}{2x+1}\, dx$.

26. (i) $\int \frac{\sqrt{x}}{\sqrt{(a^3 - x^3)}}\, dx$. (ii) $\int \frac{x^2}{\sqrt{(a^6 - x^6)}}\, dx$.

[Put $x^3 = a^3 \sin^2\theta$ in (i) and $a^3 \sin\theta$ in (ii).]

(iii) $\int \frac{dx}{x^3\sqrt{(x^2 - 1)}}$. (iv) $\int \frac{x^3\, dx}{\sqrt{(1-x^2)}}$.

27. $\int \sqrt{\frac{x}{a-x}}\, dx$. [Put $x = a\sin^2\theta$]

28. (i) $\int \frac{dx}{(a^2 - x^2)^{3/2}}$. (ii) $\int \frac{dx}{(1-x)\sqrt{(1-x^2)}}$.

29. (i) $\int (1 - \frac{1}{x^2})\, e^{x+\frac{1}{x}}\, dx$. (ii) $\int \frac{x^2+1}{(x^2-1)^2}\, dx$.

30. $\int \frac{a\cos x - b\sin x}{a\sin x + b\cos x + c}\, dx$.

31. (i) $\int \frac{dx}{x(a + b\log x)}$. (ii) $\int \frac{(\log \sec x)^2}{\cot x}\, dx$.

32. $\int \frac{x^2+1}{\sqrt[3]{(x^3+3x+6)}}\, dx$.

33. (i) $\int \cos x \cos(\sin x)\, dx$. (ii) $\int \sin x \cot^3 x\, dx$.

(iii) $\int \tan x \tan 2x \tan 3x\, dx$.

34. $\int \cos\left(2\cot^{-1}\sqrt{\frac{1-x}{1+x}}\right) dx$. [Put $x = \cos\theta$.]

35. (i) $\int \frac{\cos x}{\cos(x+\alpha)}\, dx$. (ii) $\int \frac{\cot\alpha - \cot x}{\cot\alpha + \cot x}\, dx$.

36. (i) $\int \frac{dx}{x^2(a-bx)^2}$. (ii) $\int \frac{x^7\, dx}{(1-x^4)^2}$.

37. (i) $\int \frac{x^{\frac{1}{2}}}{1 + x^{3/4}}\, dx$ [Put $x = z^4$.] (ii) $\int \frac{\sqrt{(1+x^2)}}{x^4}\, dx$.

38. (i) $\int \frac{dx}{x\sqrt{(x^4-1)}}$. (ii) $\int \frac{2x\, dx}{(1-x^2)\sqrt{(x^4-1)}}$.

[Put $x^2 = \sec\theta$.]

39. $\int \{ f(x)\,\varphi'(x) + \varphi(x)\, f'(x) \}\, dx$

40. Integrate $\frac{1}{2} f'(x)$ with respect to x^4 where

$$f(x) = \tan^{-1} x + \log\sqrt{1+x} - \log\sqrt{1-x}.$$