2nd integral $= \int \frac{(2x+2)-3}{x^2+2x+3}\, dx$

$$= \int \frac{2x+2}{x^2+2x+3}\, dx - 3\int \frac{dx}{(x+1)^2+(\sqrt{2})^2}$$

$$= \log(x^2+2x+3) - \frac{3}{\sqrt{2}} \tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right)$$

$$\therefore \quad I = x - \log(x^2+2x+3) + \frac{3}{\sqrt{2}} \tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right)$$

Note. In order to evaluate integrals of the type $\int \frac{f(x)}{ax^2+bx+c}\, dx$, (of which the above is a particular case), where $f(x)$ is a rational function of x of second or of higher degree, divide the numerator by the denominator till the numerator is of the first degree; then the integral reduces to the sumof integrals of the type $\int A_m x^m dx (m > or = 0)$ and of the type

$$\int \frac{px+q}{ax^2+bx+c}\, dx.$$

2.10 Miscellaneous Workedout Examples

Integrate the following :

Ex. 1. $\int \frac{dx}{(e^x+e^{-x})^2}$. [C. P. 1982]

Solution : $\int \frac{dx}{(e^x+e^{-x})^2} = \int \frac{e^{2x}dx}{(e^{2x}+1)^2}$

[We put $e^{2x}+1 = u \Rightarrow 2e^{2x}dx = du \Rightarrow e^{2x}dx = \frac{1}{2}du$]

$$= \frac{1}{2}\int \frac{du}{u^2} = \frac{1}{2}\left(-\frac{1}{u}\right) + c = -\frac{1}{2(e^{2x}+1)} + C$$

Ex. 2. $\int \frac{dx}{\sqrt{1-e^{-x}}}$. [C. P. 1981

]

Solution : $\int \frac{dx}{\sqrt{1-e^{-x}}}$

[We put, $1-e^{-x}=z^2 \Rightarrow e^{-x}dx = 2zdz \Rightarrow dx = \frac{2z\,dz}{e^{-x}} = \frac{2z\,dz}{1-z^2}$]

$$= \int \frac{2z\,dz}{(1-z^2)\,z} = 2\int \frac{dz}{1-z^2}$$

$$= 2 \times \frac{1}{2\cdot 1}\log\left|\frac{1+z}{1-z}\right| + C$$

$$= \log\left|\frac{1+\sqrt{1-e^{-x}}}{1-\sqrt{1-e^{-x}}}\right| + C.$$

Ex. 3. $\int \frac{dx}{x^2(2+3x)^2}$.

Solution : $\because\ 2+3x = ux,\ x = \frac{2}{u-3}$

$$\int \frac{dx}{x^2(2+3x)^2} = \int \frac{1}{\frac{4}{(u-3)^2}\times u^2 \times \frac{4}{(u-3)^2}} \times \frac{2}{(u-3)(3-u)}\cdot du$$

[We put, $2+3x = ux \Rightarrow 3dx = u\,dx + x\,du \Rightarrow (3-u)\,dx = x\,du$

$\Rightarrow dx = \frac{x\,du}{3-u}$]

$$= -\frac{1}{8}\int \frac{(u-3)^4}{u^2(u-3)^2}\cdot du$$

$$= -\frac{1}{8}\int \frac{u^2-6u+9}{u^2}\cdot du$$

$$= -\frac{1}{8}\left\{\int du - 6\int \frac{du}{u} + 9\int \frac{du}{u^2}\right\}$$

$$= -\frac{1}{8}\left\{u - 6\log|u| - \frac{9}{u}\right\} + C$$

$$= -\frac{1}{8}\left\{\frac{2+3x}{x} - \frac{9x}{2+3x} - 6\log\left|\frac{2+3x}{x}\right|\right\} + C.$$

Ex. 4. $\int \frac{\sqrt{x}\,dx}{x(x+1)}$.

Solution : $\int \frac{\sqrt{x}\,dx}{x(x+1)} = \int \frac{dx}{\sqrt{x}(x+1)} = 2\int dz = 2z + C$

$= 2\tan^{-1}\sqrt{x} + C.$

[We put $\tan^{-1}\sqrt{x} = z \Rightarrow \frac{1}{1+x}\cdot\frac{1}{2\sqrt{x}}dx = dz \Rightarrow \frac{dx}{\sqrt{x}(1+x)} 2dz$]

Ex. 5. $\int \frac{10x^9 + 10^x \cdot \log_e 10}{10^x + x^{10}}\,dx$.

Solution : Let, $10^x + x^{10} = z \Rightarrow (10x^9 + 10^x \log_e 10)dx = dz$

$\therefore \int \frac{10x^9 + 10^x \cdot \log_e 10}{10^x + x^{10}}\,dx = \int \frac{dz}{z} = \log|z| + C$

$= \log\left|\left(10^x + x^{10}\right)\right| + C.$

Ex. 6. $\int x^2 \cdot e^{x^3}\,dx$. [C. P. 1998]

Solution : $\int x^2 \cdot e^{x^3}\,dx = \frac{1}{3}\int e^t\,dt = \frac{1}{3}e^t + c = \frac{1}{3}e^{x^3} + C.$

[We put $x^3 = t \Rightarrow 3x^2 dx = dt \Rightarrow x^2 dx = \frac{1}{3}dt$]

Ex. 7. $\int \tan^3 3x\,dx$. [C. P. 1992]

Solution : $\int \tan^3 3x\,dx$

$= \int \tan^2 3x \cdot \tan 3x\,dx = \int \tan 3x(\sec^2 3x - 1)\,dx$

$= \int \tan 3x \cdot \sec^2 3x\,dx - \int \tan 3x\,dx$

$= \frac{1}{3}\int z \cdot dz - \int \frac{\sin 3x}{\cos 3x}\,dx$

[We put $\tan 3x = z \Rightarrow 3\sec^2 3x dx = dz$]

$$= \frac{1}{3} \cdot \frac{1}{2} z^2 + \frac{1}{3} \log|\cos 3x| + C$$

$$= \frac{1}{6} \cdot \tan^2 3x + \frac{1}{3} \log|\cos 3x| + C.$$

Ex. 8. $\int \frac{dx}{x^2 (x^4 + 1)^{\frac{3}{4}}}$. [C. P. 1992]

Solution : $\int \frac{dx}{x^2 (x^4 + 1)^{\frac{3}{4}}}$

$$\therefore \quad I = \int \frac{dx}{x^2 \left\{x^4 \left(1 + \frac{1}{x^4}\right)\right\}^{\frac{3}{4}}} = \int \frac{dx}{x^2 \cdot x^3 \left(1 + \frac{1}{x^4}\right)^{\frac{3}{4}}}$$

$$= \int \frac{dx}{x^5 \left(1 + \frac{1}{x^4}\right)^{\frac{3}{4}}} \quad \ldots (1)$$

Let us substitute, $1 + \frac{1}{x^4} = t \Rightarrow \frac{-4}{x^5} dx = dt \Rightarrow \frac{dx}{x^5} = -\frac{1}{4} dt$

From (1)

$$I = -\frac{1}{4} \int \frac{dt}{t^{\frac{3}{4}}} = -\frac{1}{4} \int t^{-\frac{3}{4}} dt = -\frac{1}{4} \cdot 4t^{\frac{1}{4}} + C$$

$$= -\left(1 + \frac{1}{x^4}\right)^{\frac{1}{4}} + c = -\frac{1}{x}(1 + x^4)^{\frac{1}{4}} + C.$$

Ex. 9. $\int \frac{x\,dx}{x^4 - x^2 - 2}$. [C. P. 1996]

Solution : $\int \frac{x\,dx}{x^4 - x^2 - 2} = \frac{1}{2} \int \frac{dz}{z^2 - z - 2}$

[We put $x^2 = z \Rightarrow 2xdx = dz \Rightarrow xdx = \frac{1}{2}dz$]

$$= \frac{1}{2}\int \frac{dz}{\left(z^2 - \frac{1}{2}\right)^2 - \left(\frac{3}{2}\right)^2}$$

$$= \frac{1}{2}\cdot\frac{1}{2\cdot\frac{3}{2}}\cdot\log\left|\frac{z - \frac{1}{2} - \frac{3}{2}}{z - \frac{1}{2} + \frac{3}{2}}\right| + C$$

$$= \frac{1}{6}\log\left|\frac{x^2 - 2}{x^2 + 1}\right| + C.$$

Ex. 10. $\int x^{\frac{13}{2}}\left(1 + x^{\frac{5}{2}}\right)^{\frac{1}{2}} dx$.

Solution : Let us substitute, $1 + x^{\frac{5}{2}} = t^2 \Rightarrow \frac{5}{2}x^{\frac{3}{2}}dx = 2t\,dt$

$$\Rightarrow x^{\frac{3}{2}}dx = \frac{4}{5}t\,dt$$

$$\therefore \quad x^{\frac{5}{2}} = t^2 - 1, \; x^5 = \left(t^2 - 1\right)^2$$

$$\int x^{\frac{13}{2}}\left(1 + x^{\frac{5}{2}}\right)^{\frac{1}{2}} dx = \int\left(1 + x^{\frac{5}{2}}\right)^{\frac{1}{2}}\cdot x^5\cdot x^{\frac{3}{2}}\cdot dx$$

$$= \int t\cdot\left(t^2 - 1\right)^2\cdot\frac{4}{5}t\,dt = \frac{4}{5}\int\left(t^6 - 2t^4 + t^2\right)dt$$

$$= \frac{4}{5}\left\{\frac{1}{7}t^7 - \frac{2}{5}t^5 + \frac{1}{3}t^3\right\} + c = \frac{4}{5}t^3\left(\frac{1}{7}t^4 - \frac{2}{5}t^2 + \frac{1}{3}\right) + C$$

$$= \frac{4}{5}\left(1 + x^{\frac{5}{2}}\right)^{\frac{3}{2}}\left\{\frac{1}{7}\left(1 + x^{\frac{5}{2}}\right)^2 - \frac{2}{3}\left(1 + x^{\frac{5}{2}}\right) + \frac{1}{3}\right\} + C.$$

Ex. 11. $\int \frac{x^2-1}{x^4-x^2-1} dx$. [*C. P. 1999*]

Solution : Let, $I = \int \frac{x^2-1}{x^4-x^2-1} dx$

$$= \int \frac{(x^2-1)}{x^2\left(x^2-1+\frac{1}{x^2}\right)} dx$$

$$= \int \frac{\left(1-\frac{1}{x^2}\right)}{\left(x+\frac{1}{x}\right)^2-3} dx$$

Now, we substitute, $x+\frac{1}{x}=z \Rightarrow \left(1-\frac{1}{x^2}\right)dx = dz$

So, $I = \int \frac{dz}{z^2-(\sqrt{3})^2} = \frac{1}{2\sqrt{3}} \log\left|\frac{z-\sqrt{3}}{z+\sqrt{3}}\right| + C$

$$= \frac{1}{2\sqrt{3}} \log\left|\frac{x+\frac{1}{x}-\sqrt{3}}{x+\frac{1}{x}+\sqrt{3}}\right| + c = \frac{1}{2\sqrt{3}} \log\left|\frac{x^2-\sqrt{3}x+1}{x^2+\sqrt{3}x+1}\right| + C.$$

Ex. 12. $\int \sqrt{\tan x}\, dx$.

Solution : $\int \sqrt{\tan x}\, dx = \int \frac{\sqrt{\tan x}\cdot\sec^2 x}{1+\tan^2 x}\cdot dx = \int \frac{2t^2 dt}{1+t^4}$

[We put $\tan x = t^2 \Rightarrow \sec^2 x\, dx = 2t\, dt$]

$$= \int \frac{2\,dt}{\frac{1}{t^2}+t^2} = \int \frac{\left(1-\frac{1}{t^2}\right)+\left(1+\frac{1}{t^2}\right)}{t^2+\frac{1}{t^2}}\cdot dt.$$