REV-01
MSM/24/29

2023/06

SET A

M.Sc. MATHEMATICS
SECOND SEMESTER
ABSTRACT ALGEBRA II
MSM – 203
[USE OMR FOR OBJECTIVE PART]

Duration: 1:30 hrs. **Full Marks: 35**

(Objective)

Time: 15 mins. Marks: 10

Choose the correct answer from the following: ***1×10=10***

1. Which of the following is/are true?
   a. The unit group $U(n)$ is a nilpotent group
   b. The unit group $U(n)$ is a nilpotent group iff n is prime
   c. The unit group $U(n)$ is not a nilpotent group
   d. The unit group $U(n)$ is a nilpotent group for some n

2. Let p, q be distinct primes. Then
   a. $\frac{\mathbb{Z}}{p^2q\mathbb{Z}}$ has exactly 3 distinct ideals.
   b. $\frac{\mathbb{Z}}{p^2q\mathbb{Z}}$ has exactly 3 distinct prime ideals.
   c. $\frac{\mathbb{Z}}{p^2q\mathbb{Z}}$ has exactly 2 distinct prime ideals.
   d. $\frac{\mathbb{Z}}{p^2q\mathbb{Z}}$ has unique maximal ideals.

3. Which of the following is/are unique factorization domain:
   a. $\mathbb{Z}_p[x]$, where p is prime
   b. $\mathbb{Z}[i]$
   c. $\mathbb{Z}[x]$
   d. All the above

4. Which of the following is/are true
   a. $S_n, n \geq 5$ is a solvable group but not Nilpotent
   b. $S_n, n \geq 5$ is a Nilpotent group but not solvable
   c. $S_n, n \geq 5$ is both solvable and Nilpotent group
   d. $S_n, n \geq 5$ is neither solvable nor Nilpotent group

5. Let F be a field with non-zero characteristic, then
   a. F has a subfield isomorphic to $\mathbb{Q}$.
   b. F has a subfield isomorphic to $\mathbb{Z}_p$ for prime p.
   c. F has a subfield isomorphic to either $\mathbb{Z}_p$ or $\mathbb{Q}$.
   d. None of these

6. Which of the following is/are true
   a. 7 is prime in the ring $\mathbb{Z}[\sqrt{5}]$.
   b. 7 is unit in the ring $\mathbb{Z}[\sqrt{5}]$.
   c. 7 is irreducible in the ring $\mathbb{Z}[\sqrt{5}]$.
   d. All the above

7. Given a polynomial $f(x) = a_0 + a_1x + \cdots + a_nx^n$, where a_i's are integers, then content of $f(x)$ is

   a. $\gcd(a_0, a_1, \cdots, a_n)$

   b. $\mathrm{lcm}(a_0, a_1, \cdots, a_n)$

   c. Mean of $a_0, a_1, \cdots, a_n$

   d. None of these

8. $\frac{\mathbb{Z}_2[x]}{<x^3+x^2+1>}$ is

   a. A field having 8 elements

   b. A field having 9 elements

   c. An infinite field

   d. Not a field

9. Which of the following statement is/are not necessarily true?

   a. A group of order 4 is solvable

   b. A group of order 25 is solvable.

   c. A group of order 21 is solvable.

   d. All the above.

10. If $\mathbb{Z}[i]$ is the ring of Gaussian integers, the quotient $\frac{\mathbb{Z}[i]}{<3-i>}$ is isomorphic to

    a. $\mathbb{Z}$

    b. $\frac{\mathbb{Z}}{3\mathbb{Z}}$

    c. $\frac{\mathbb{Z}}{4\mathbb{Z}}$

    d. $\frac{\mathbb{Z}}{10\mathbb{Z}}$

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Descriptive

Time : 1 hr. 15 mins. Marks : 25

[Answer question no.1 & any two (2) from the rest]

1. Show that 3+2=5

 a. $f(x) = \frac{3}{7}x^4 - \frac{2}{7}x^2 + \frac{9}{35}x + \frac{3}{5}$ is irreducible over $\mathbb{Q}$.

 b. $f(x) = 3x^5 + 15x^4 - 20x^3 + 10x + 20$ is irreducible over $\mathbb{Q}$.

2. a. Find all the composition series of $\mathbb{Z}_{30}$ and show they are equivalent. 4+4+2 =10

 b. Show that the quaternion group Q_8 is a Nilpotent group. Is Q_8 solvable?

 c. Prove that – Any non abelian simple group is not solvable.

3. a. Prove that $< 2 + 2i >$ is not a prime ideal of $\mathbb{Z}[i]$. 4+3+3 =10

 b. Using Fundamental theorem of ring homomorphism show that $\frac{\mathbb{Z}[x]}{<x>}$ is not a field.

 c. Determine all ring homomorphisms from $\mathbb{Z}_{30}$ to $\mathbb{Z}_{20}$.

4. a. Construct a field of order 27. 3+4+3 =10

 b. In $\mathbb{Z}[\sqrt{-5}]$, prove that $1 + 3\sqrt{-5}$ is irreducible but not prime.

 c. In $\mathbb{Z}[\sqrt{-6}]$, show that 10 does not factor uniquely as a product of irreducible

5. a. Prove that the group $(\mathbb{Z}, +)$ has no composition series. 3+3+4 =10

 b. Prove that – For any prime p, the pth cyclotomic polynomial

$$x^{p-1} + x^{p-2} + \cdots + x + 1$$

 is irreducible over $\mathbb{Q}$.

 c. Let d be a function from the nonzero elements of $\mathbb{Z}$ to the nonnegative integers. Show that – The ring $\mathbb{Z}$ is a Euclidean domain with $d(a) = |a|$. Is $\mathbb{Z}$ a UFD (unique factorization domain)? Justify your answer.

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