REV-01
MSM/01/05

2024/05

SET A

M.SC. MATHEMATICS
FOURTH SEMESTER
ADVANCED ALGEBRA
MSM - 403C

[USE OMR FOR OBJECTIVE PART]

Duration: 1:30 hrs. **Full Marks: 35**

(Objective)

Time: 15 mins. **Marks: 10**

Choose the correct answer from the following: ***1×10=10***

1. Let P be a property of a free group. Then P is true if P denotes the property
   - **a.** Every free group is torsion free.
   - **b.** A free group on $X = \{x, y\}$ is abelian.
   - **c.** A free group on $X = \{x\}$ is finite cyclic.
   - **d.** A free group on $X = \{x\}$ is infinite cyclic.

2. For any non-empty set $X = \{x_\lambda : \lambda \in \Lambda\}$ every element of $X \cup X^{-1}$ is a word of length
   - **a.** greater than 1
   - **b.** greater than 2
   - **c.** equal to 0
   - **d.** equal to 1

3. An abelian group G is said to be free abelian if there is a set $X \subset G$ such that
   - **a.** X is linearly independent
   - **b.** X generates G
   - **c.** Both (a) and (b) hold
   - **d.** None of (a) and (b) holds

4. $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ is a free abelian group having its basis as
   - **a.** $\{(1,0,0),(0,2,0),(0,0,3)\}$
   - **b.** $\{(1,0,0),(2,1,0),(0,0,1)\}$
   - **c.** $\{(2,0,0),(0,1,0),(0,0,1)\}$
   - **d.** $\{(4,0,0),(0,3,0),(0,0,2)\}$

5. Let $= \{,\}$. The word $^{-1}$ is adjacent to
   - **a.** $^{-1-1}$
   - **b.** $^{-1-1}$
   - **c.** $yyyxy^{-1}$
   - **d.** 1

6. Let $X = \{x, y\}$. The word $xy^{-1}x$ is equivalent to
   - **a.** $xxy^{-1}y^{-1}x$
   - **b.** $xxx^{-1}y^{-1}yy^{-1}x$
   - **c.** $xyy^{-1}xx^{-1}x$
   - **d.** $xyxyxy^{-1}x$

7. Let A be any additive abelian group. Then A is
   - **a.** a left $\mathbb{R}$-module, $\mathbb{R}$ being the ring of real numbers.
   - **b.** a left $\mathbb{Q}$-module, $\mathbb{Q}$ being the ring of rational numbers.
   - **c.** a left $\mathbb{Z}$-module, $\mathbb{Z}$ being the ring of integers.
   - **d.** None of the above.

8. Let R be a subring of a ring S. Then
   - **a.** R is a left (right) S module.
   - **b.** S is a left (right) R module.
   - **c.** Zero subrings (0) is not a left (right) S-module.
   - **d.** None of the above.

9. If M is an irreducible unital R-module then for any $x \in M$

a. $Rx \subset M$

b. $M \subset Rx$

c. $Rx = M$

d. None of these

10. The submodules of quotient module $\frac{M}{N}$ are of the form $\frac{U}{N}$ where U is a submodule of

a. M

b. N

c. M containing N

d. N containing M

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Descriptive

Time : 1 hr. 15 mins. Marks: 25

[Answer question no.1 & any two (2) from the rest]

1. If $\{G_\lambda : \lambda \in \wedge\}$ be any families of groups then define direct product $\prod_{\lambda\in\wedge} G_\lambda$ where 5
   (i) $\wedge = \{1, 2, 3, \cdots, n\}$
   (ii) $\wedge$ is an arbitrary set of indices.

2. a. Let the group G have subgroups G_i for $1 \leq i \leq n$ such that 5+5=10
   (i) G_i commutes with G_j elementwise for $1 \leq i < j \leq n$
   (ii) $G = G_1 G_2 G_3 \cdots G_n$
   (iii) $G_i \cap G_1 G_2 \cdots G_{i-1} G_{i+1} \cdots G_n = \{1\}$ for $1 \leq i \leq n$
   Then prove that $G \cong G_1 \times G_2 \times \cdots \times G_n$.

   b. Show that $\mathbb{Z} \times \mathbb{Z}$ is free abelian group where $\mathbb{Z}$ is the additive groups of integers.

3. a. If G is a non-zero free abelian group with a basis of r elements, then prove that G is isomorphic to $\mathbb{Z} \times \mathbb{Z} \times \cdots \times \mathbb{Z}$ for r factors. 5+5=10

   b. How do you identify two words u and v as equivalent in $X \cup X^{-1}$ where X is any nonempty set? Show that the equivalence of words in $X \cup X^{-1}$ is an equivalence relation.

4. a. For any ring R define a left R-module as well as a right R-module. When is a left R-module also a right module? What are unital R-modules? 5+5=10

   b. What is a cyclic submodule generated by an element x in an R-module M? If an R-module M be generated by a set $\{x_1, x_2, \cdots, x_n\}$ and $1 \in R$ then show that

$$M = \sum_{i=1}^{n} Rx_i = \{r_1x_1 + r_2x_2 + \cdots + r_nx_n : r_i \in R\}$$

5. **a.** Let M be an R-module. Show that 5+5=10

$$RM = \left\{\sum r_i m_i : r_i \in R, m_i \in M, i = 1, 2, \cdots, n\right\}$$

is an R-submodule of the R-module M. When is an R-module M called irreducible?

b. Define homomorphism from an R-module M into another R-module N where R is a ring. If f be an R-homomorphism of a R-module M into an R-module N then show that

$$\frac{M}{\ker f} \cong f(M)$$

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